Modelling, Uncertainty and Data for Engineers (MUDE)

Signal Processing: discrete time sampling, DFT & spectrum

Christian Tiberius

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Signal Processing: sampling

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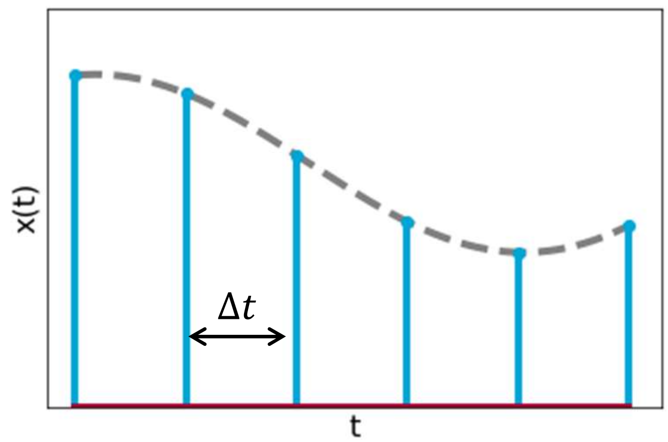
Discrete signals

A **discrete** signal has a value only at discrete values of the running variable (usually time). Formally the signal is then to be referred to as discrete-time signal.

The interval between these discrete values of running variable is often uniform, e.g. Δt.

In-between these values, signal may be zero, undefined, or of no interest!



Note:
continuous-time signal is written as $x(t)$
discrete-time signal is usually written as $x[n]$, or x_n (sequence x_0, x_1, x_2 ...)

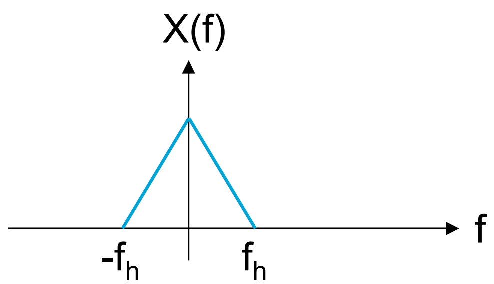
What about discrete time (sampled) signals?
Does sampling have any impact on $X(f)$?

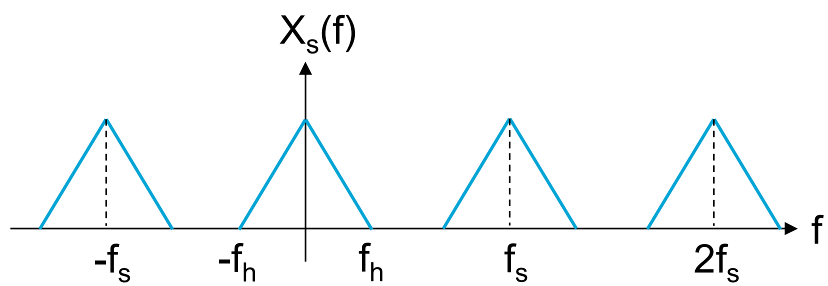
Fourier transform of sampled signal

Finally, Fourier transform of *sampled* signal becomes:

$$X_s(f) = \sum_{k=-\infty}^{k=\infty} X(f - kf_s)$$

so, spectrum of sampled signal is spectrum of original signal, but repeated with 'period' f_s (in frequency domain); copies of spectrum are called **aliases**





(left) spectrum of an assumed original signal $x(t)$, (right) and that of its sampled equivalent

Sampling theorem

Band-limited signal $x(t)$, having no frequency components above f_h Hertz, is *completely specified* by samples taken at *uniform* rate greater than $2f_h$ Hertz.

frequency $2f_h$ is called **Nyquist rate**

Note: Nyquist *rate* is characteristic of *signal*, whereas Nyquist *frequency*, $\frac{f_s}{2}$, is characteristic of *sampling system*

(in practice we consider only domain $-\frac{f_s}{2} < f \leq \frac{f_s}{2}$ of spectrum obtained from sampled signal)

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Aliasing

Note that in order to reconstruct original continuous-time signal from samples, it is crucial to sample signal at rate larger than Nyquist rate.

When sampling signal *below* this rate, the adjacent spectra (aliases) will overlap, and it will be impossible to reconstruct signal from its samples. *)

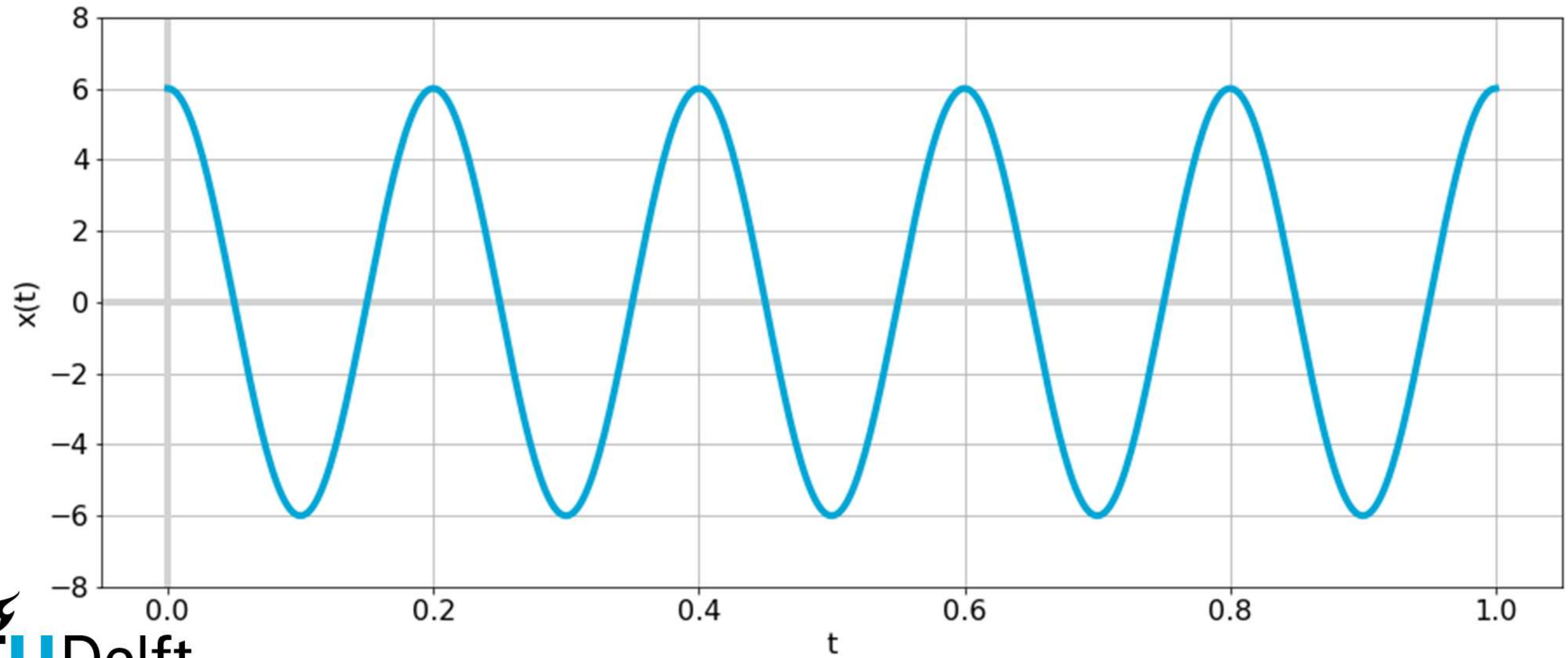
This is called **aliasing**, and is illustrated in the following example.

*) information is lost, and this is irreversible … :-(

Aliasing example – signal

As example we study the effect of sampling sinusoidal signal with frequency of $f_c = 5$ Hz

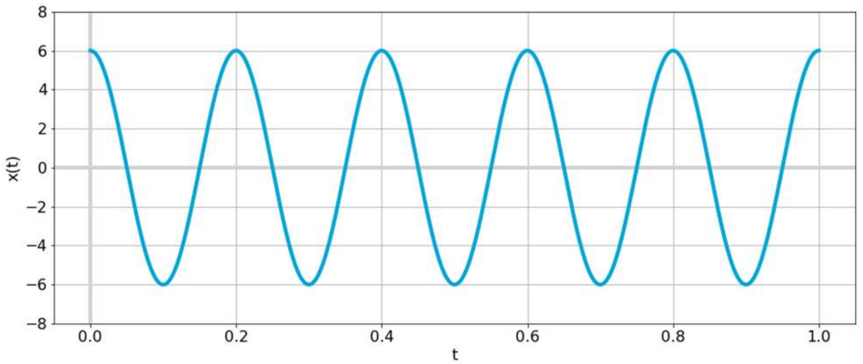
$$x(t) = 6\cos(10\pi t)$$



Aliasing example – signal

First, we look at *correctly* sampled signal, with $f_s = 14$ Hz ($f_s > 2f_c$)

Spectrum (which is real, because $x(t)$ is even) of original continuous-time signal will have two Dirac-functions with weight 3, at $f = 5$ Hz and $f = -5$ Hz, i.e. $\mathrm{X}(f) = \frac{6}{2}[\delta(f-5) + \delta(f+5)]$



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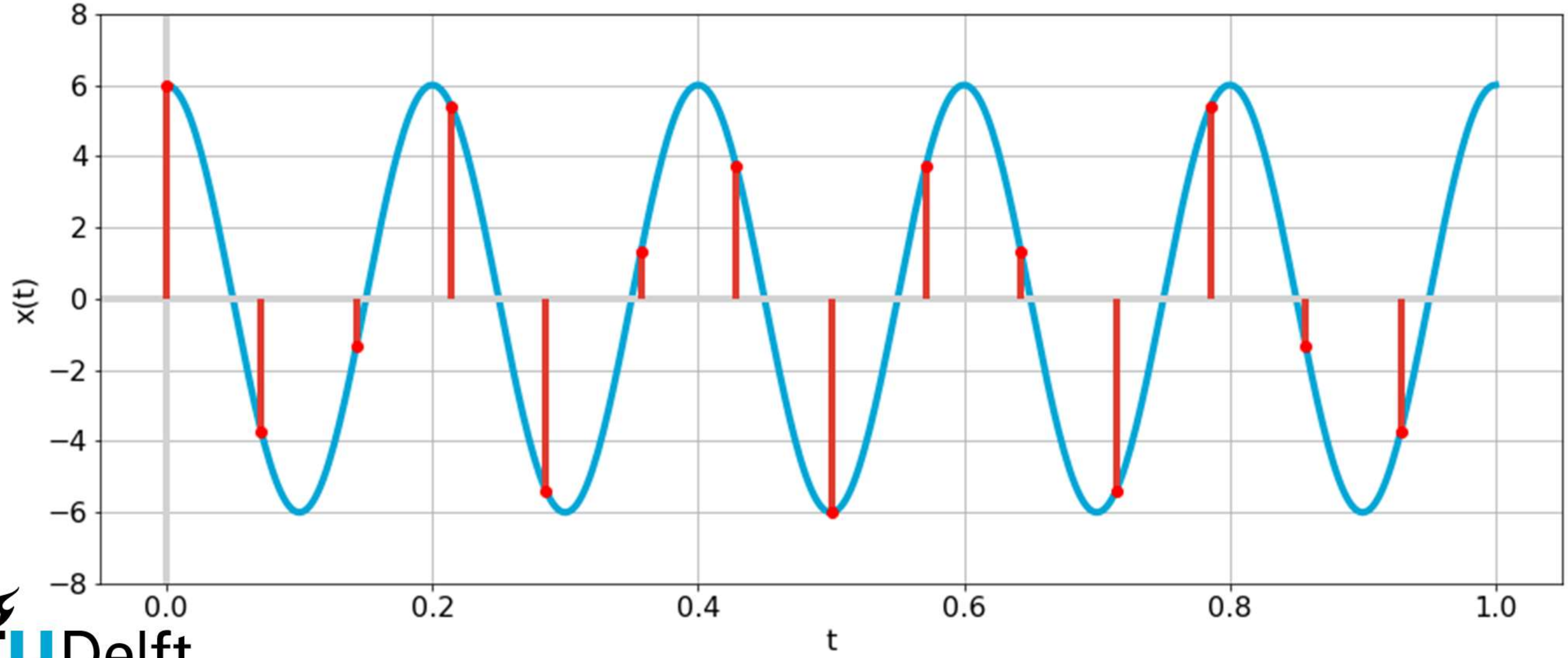
Aliasing example – sampled at 14 Hz

$$x(t) = 6\cos(10\pi t)$$



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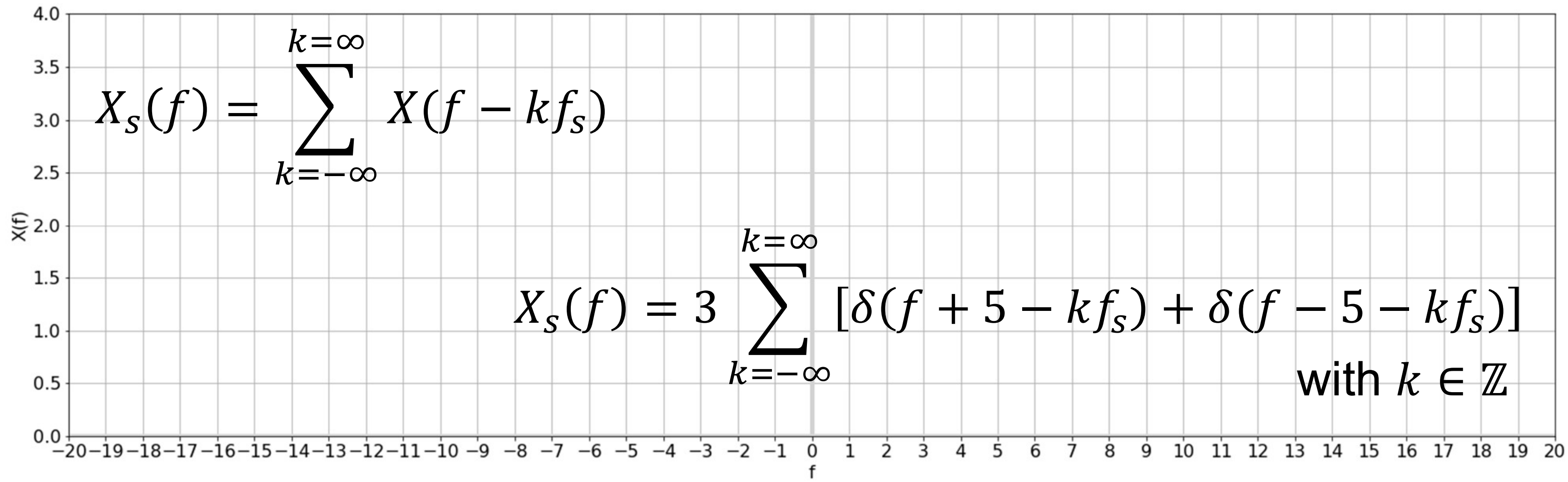
Aliasing example – spectrum

$$x(t) = 6\cos(10\pi t)$$

$$f_s = 14 \text{ Hz}$$



$$X_s(f) = \sum_{k=-\infty}^{k=\infty} X(f - kf_s)$$

$$X_s(f) = 3 \sum_{k=-\infty}^{k=\infty} [\delta(f + 5 - kf_s) + \delta(f - 5 - kf_s)]$$

with $k \in \mathbb{Z}$

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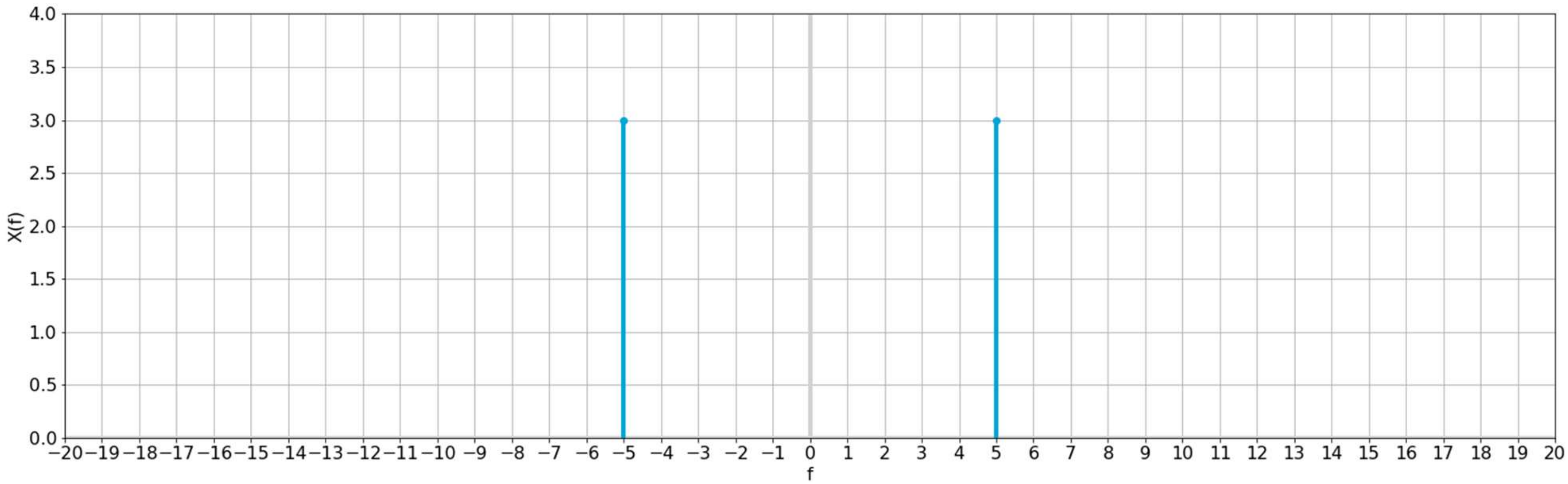
Aliasing example – spectrum

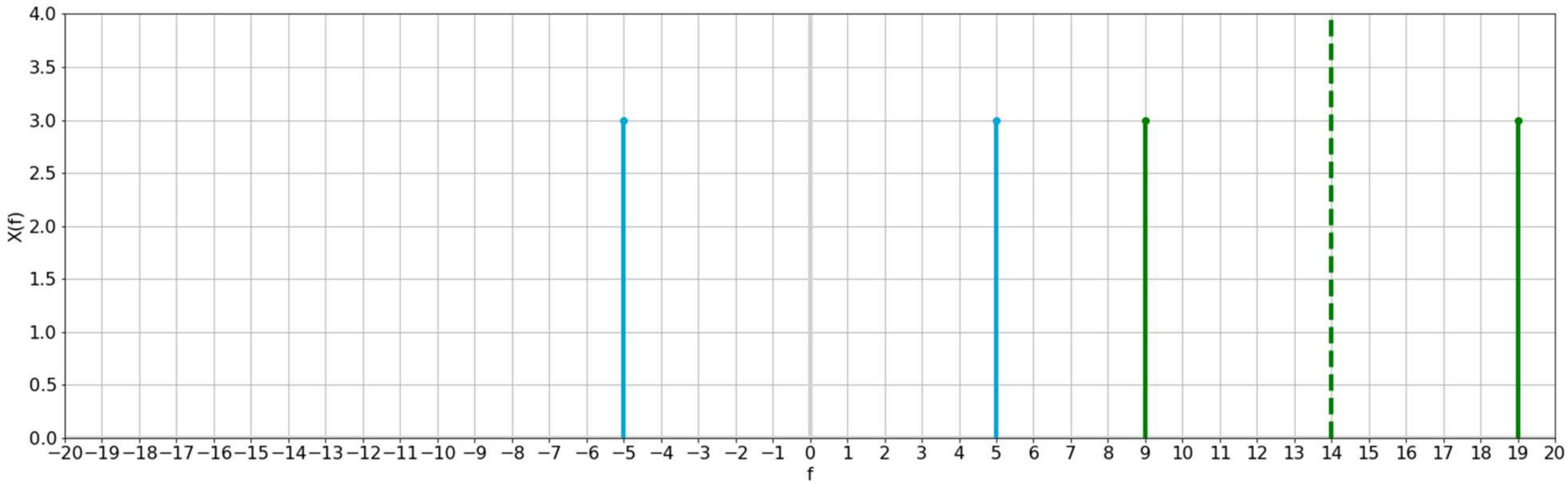
$$x(t) = 6\cos(10\pi t)$$
$$f_s = 14 \text{ Hz}$$



$$k = 0$$

Aliasing example – spectrum

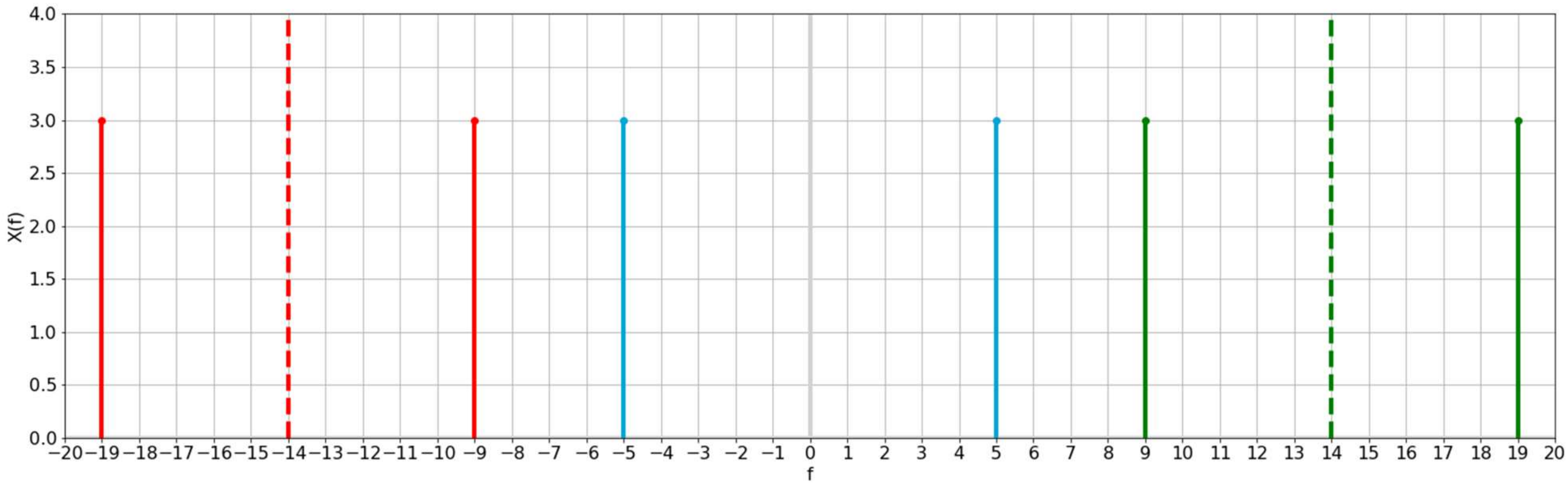
$$x(t) = 6\cos(10\pi t)$$
$$f_s = 14 \text{ Hz}$$



$k = 0$ $\qquad$ $k = 1$

Aliasing example – spectrum

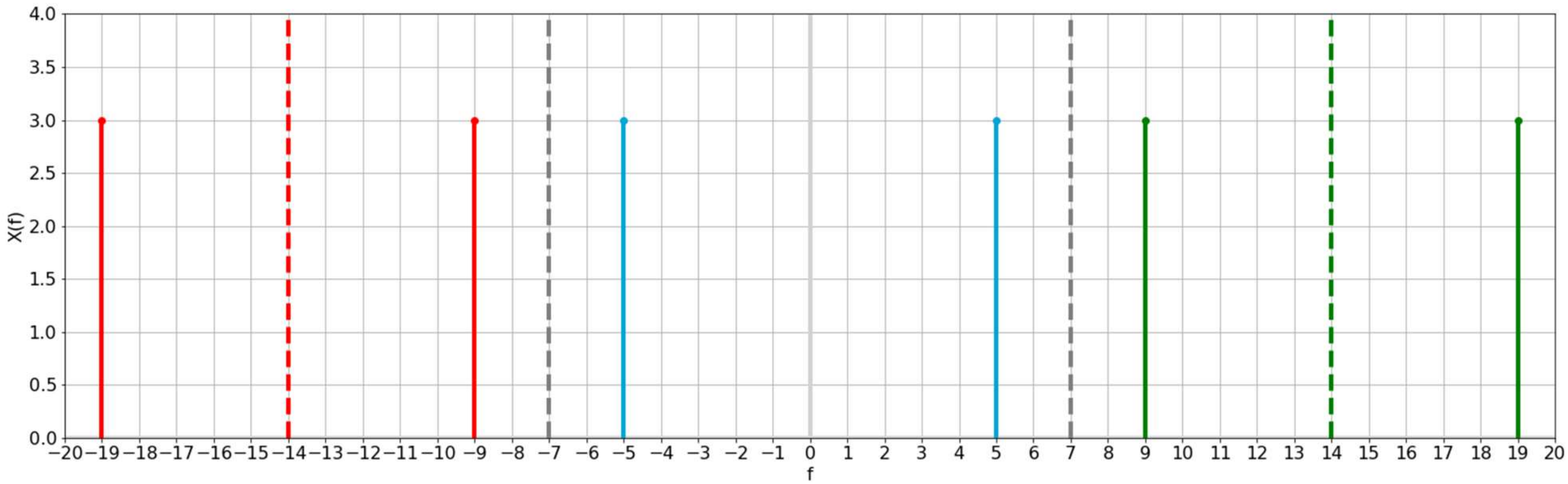
$$x(t) = 6\cos(10\pi t)$$
$$f_s = 14 \text{ Hz}$$



$k = -1$ $\qquad$ $k = 0$ $\qquad$ $k = 1$

Aliasing example – spectrum

$$x(t) = 6\cos(10\pi t)$$
$$f_s = 14 \text{ Hz}$$



we consider only domain $-\frac{f_s}{2} < f \leq \frac{f_s}{2}$

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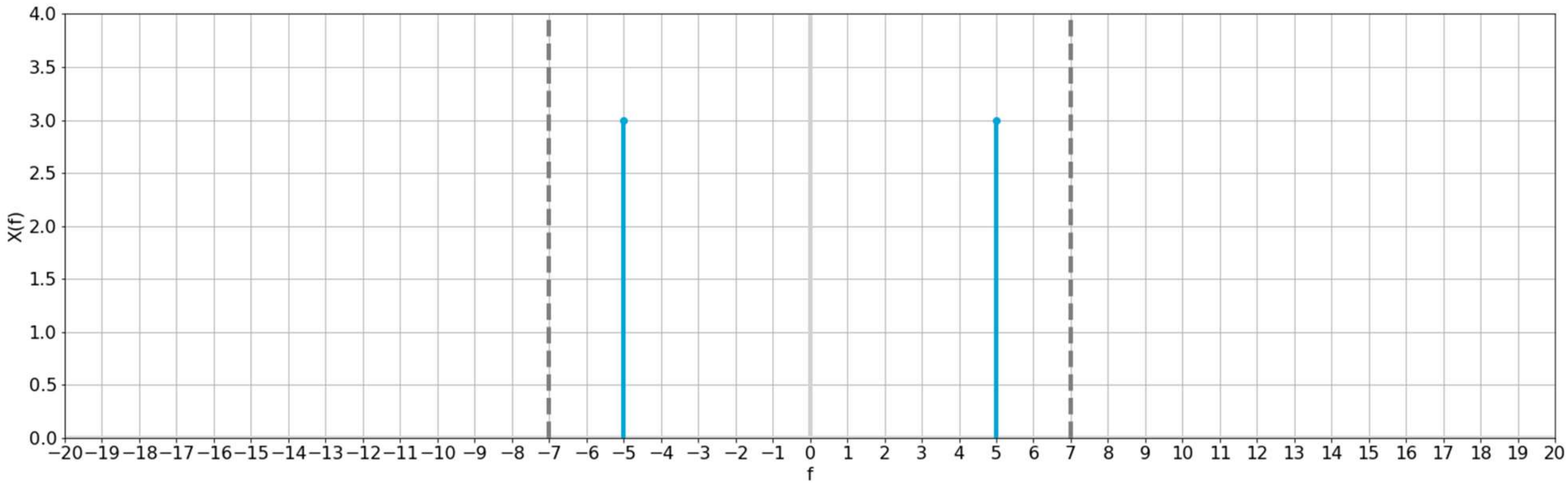
Aliasing Example – Spectrum

$$x(t) = 6\cos(10\pi t)$$
$$f_s = 14 \text{ Hz}$$



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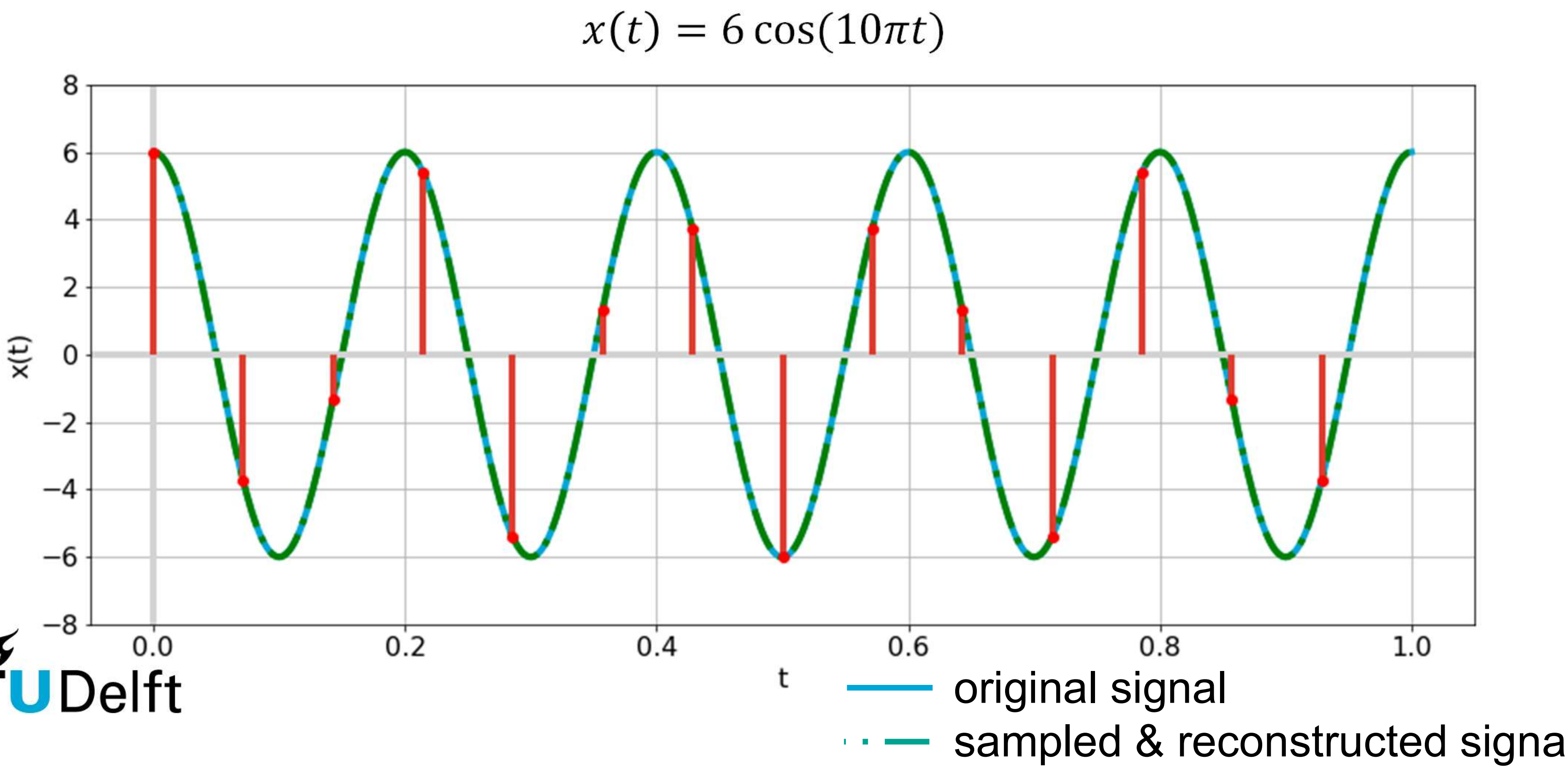
Aliasing example – correct result



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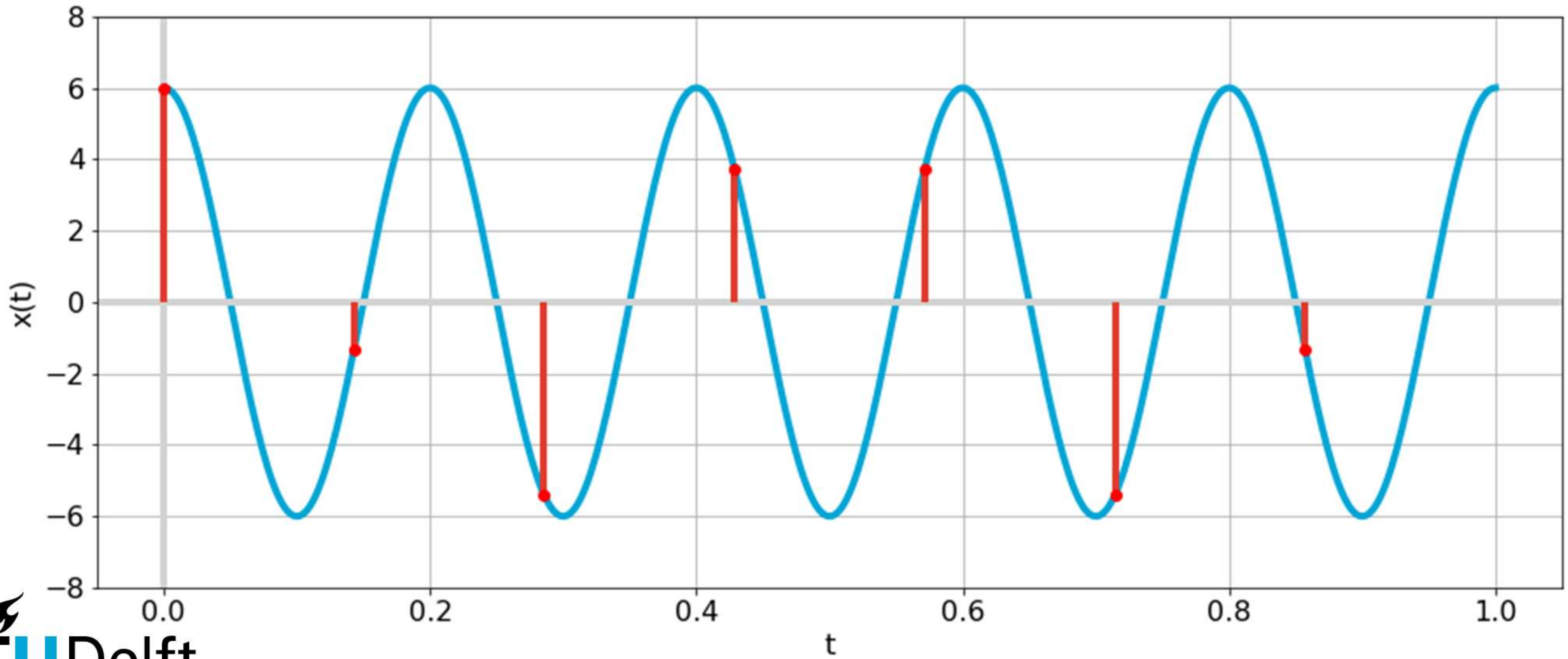
Aliasing example – sampled at 7 Hz

violating sampling theorem:

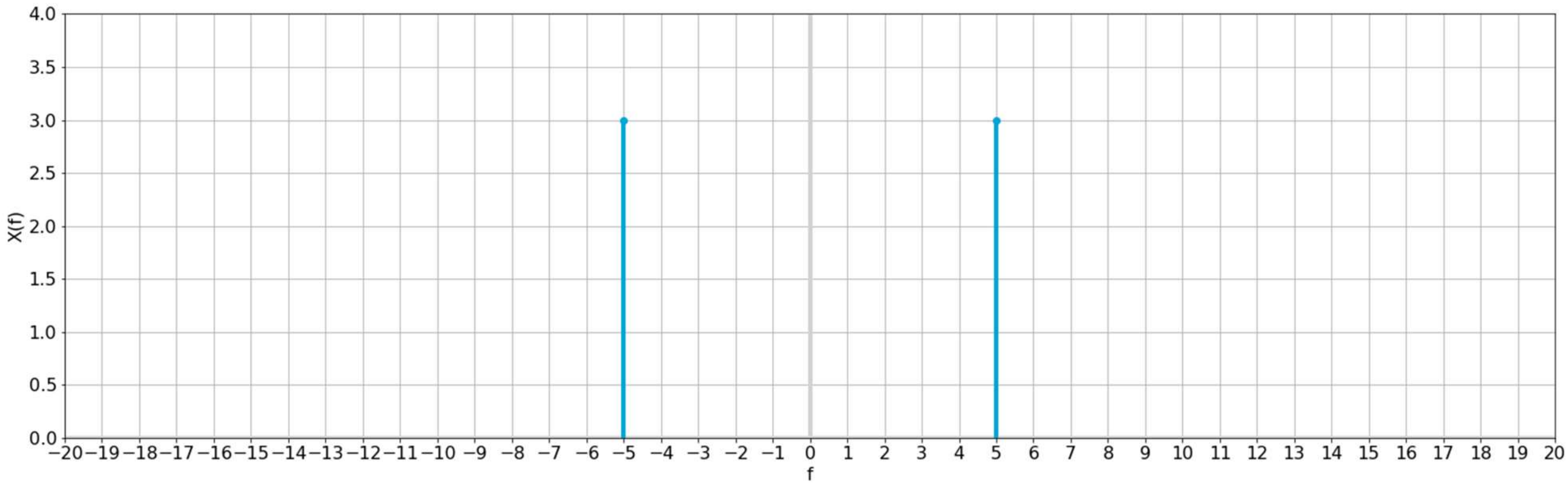
$f_s = 7$ Hz ($f_s \ngtr 2f_c$)

$$x(t) = 6\cos(10\pi t)$$



Aliasing example – spectrum

$$x(t) = 6\cos(10\pi t)$$
$$f_s = 7 \text{ Hz}$$



$$k = 0$$

Aliasing example – spectrum

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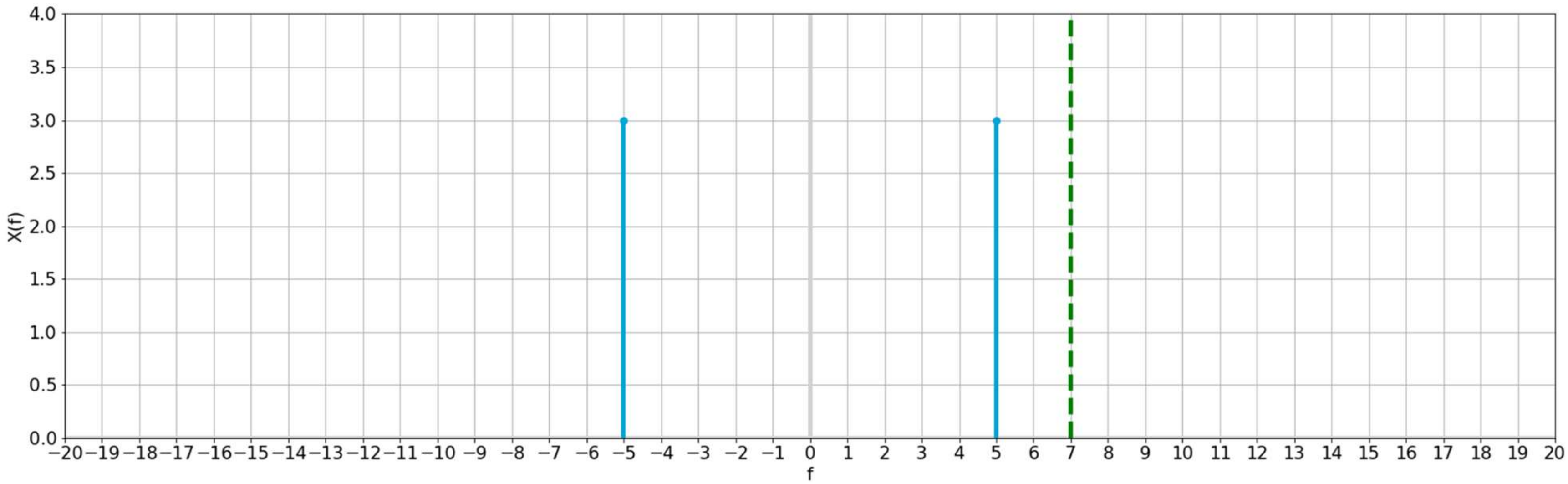

$k = 0$ $\qquad$ $k = 1$

Aliasing example – spectrum

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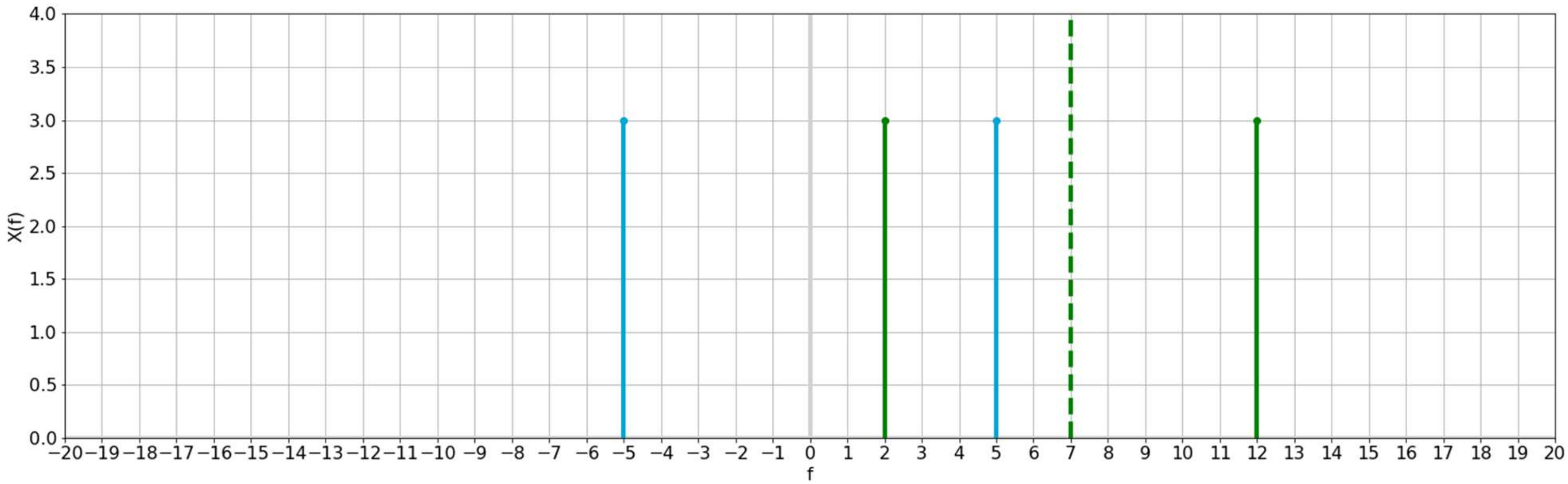

$$k = 0 \qquad k = 1$$

Aliasing example – spectrum

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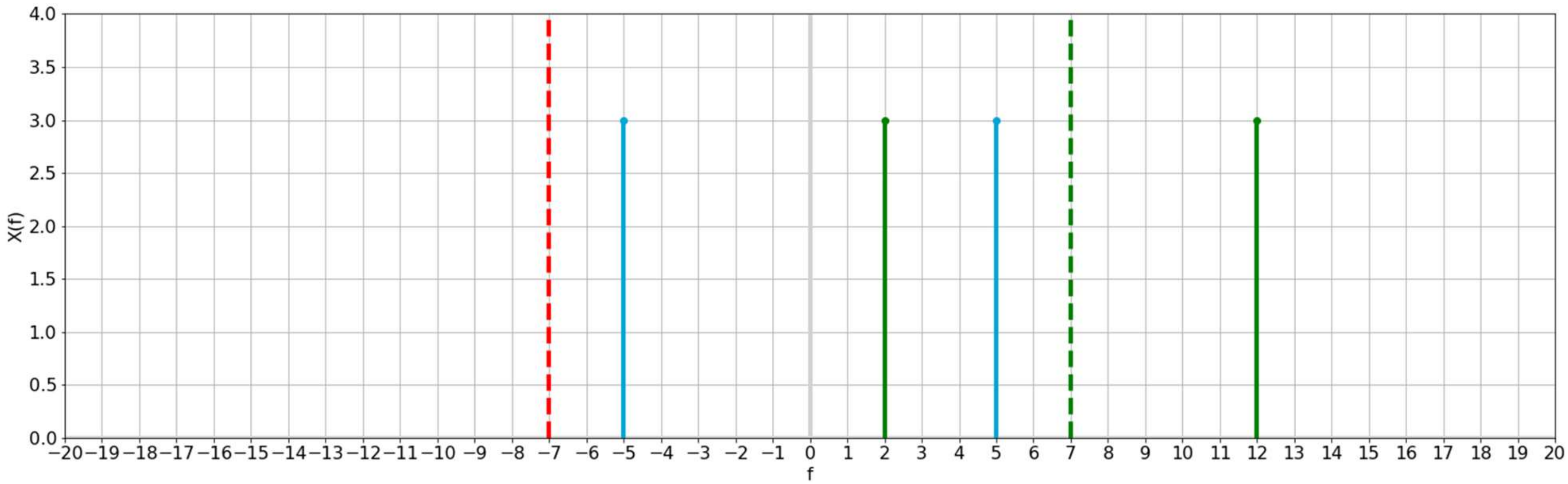

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Aliasing example – spectrum

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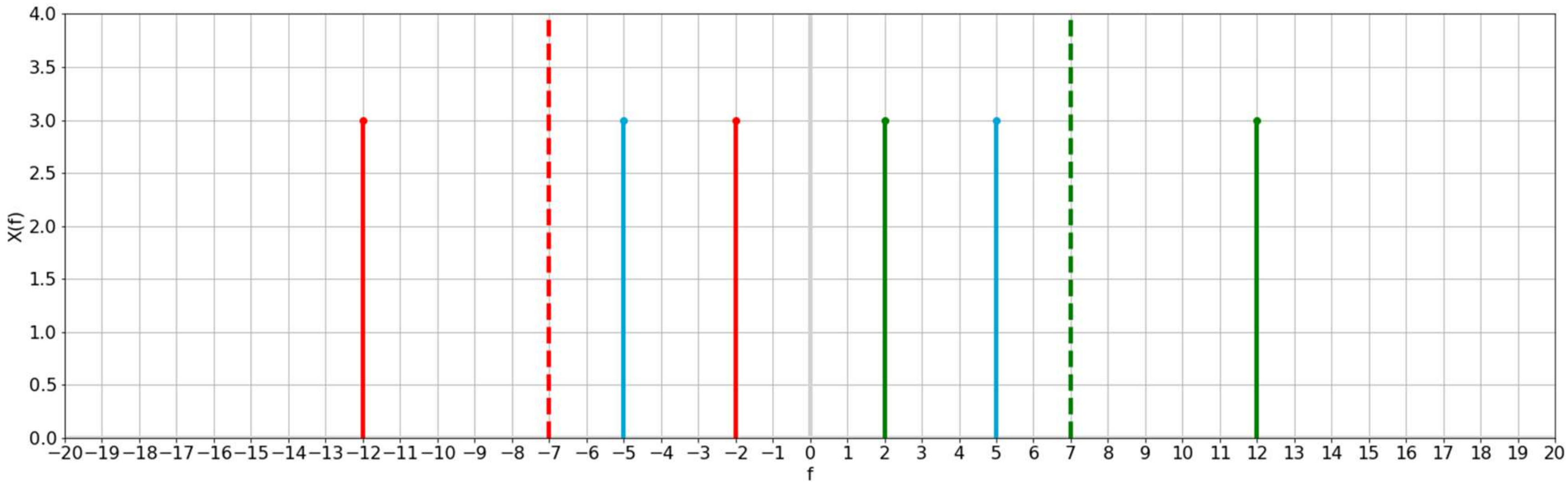

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Aliasing example – spectrum

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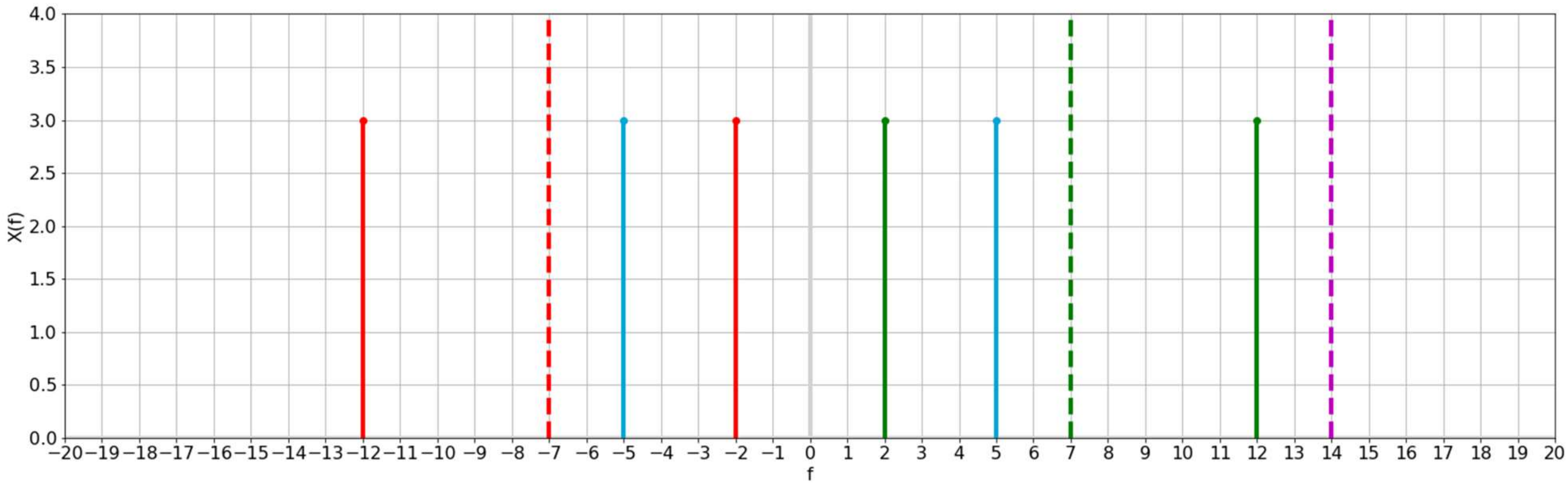

$k = -1$ $\qquad k = 0$ $\qquad k = 1$ $\qquad k = 2$

Aliasing example – spectrum

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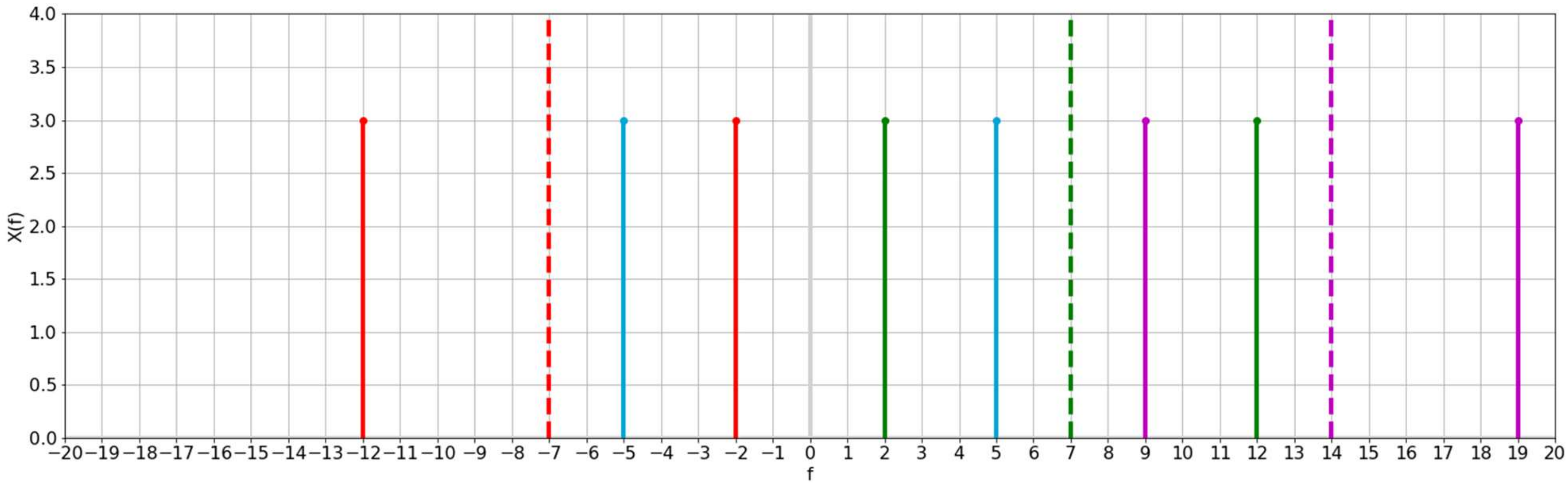

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Aliasing example – spectrum

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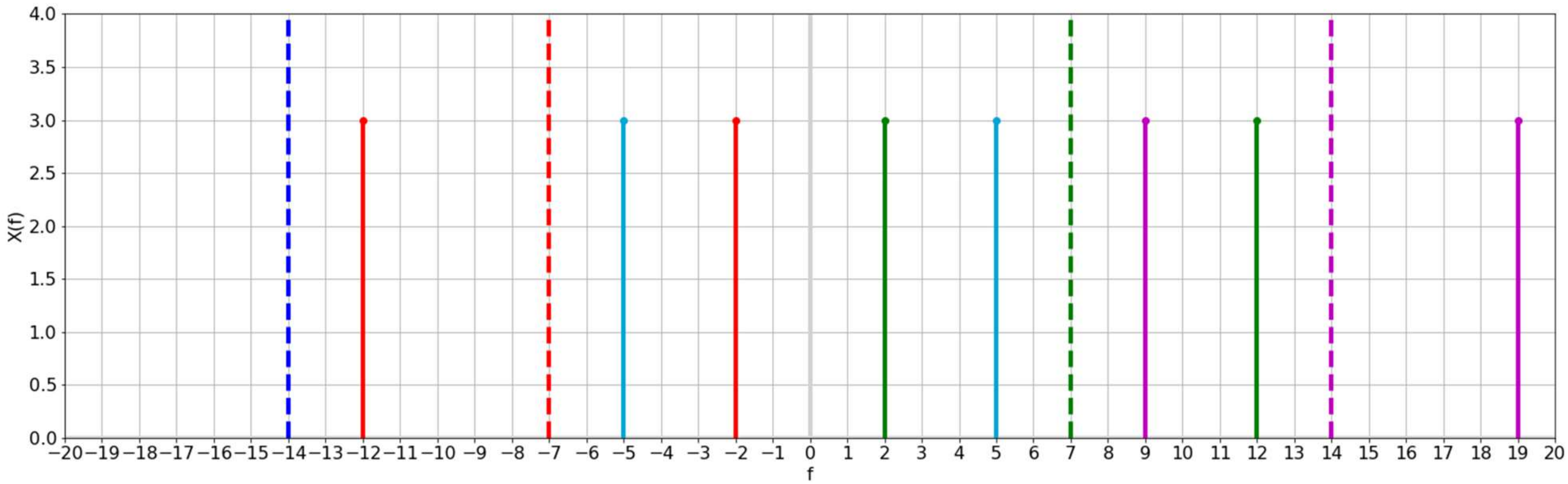

$k = -2$ $\quad k = -1$ $\quad k = 0$ $\quad k = 1$ $\quad k = 2$

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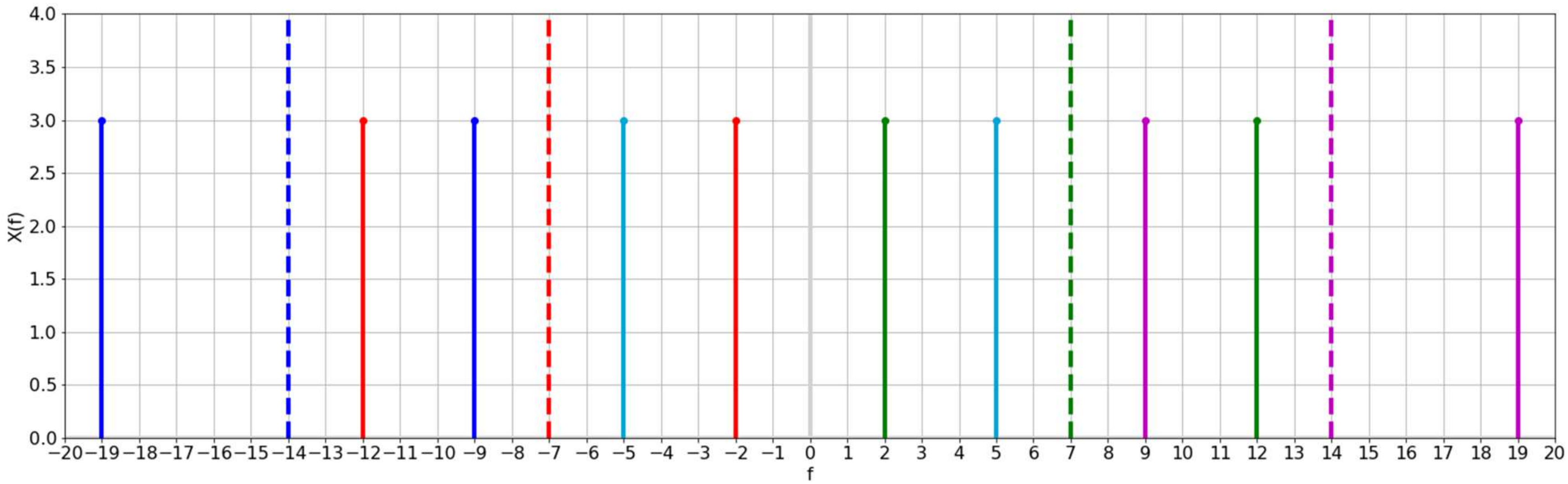
Aliasing example – spectrum

$$x(t) = 6\cos(10\pi t)$$
$$f_s = 7 \text{ Hz}$$



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Aliasing example – spectrum

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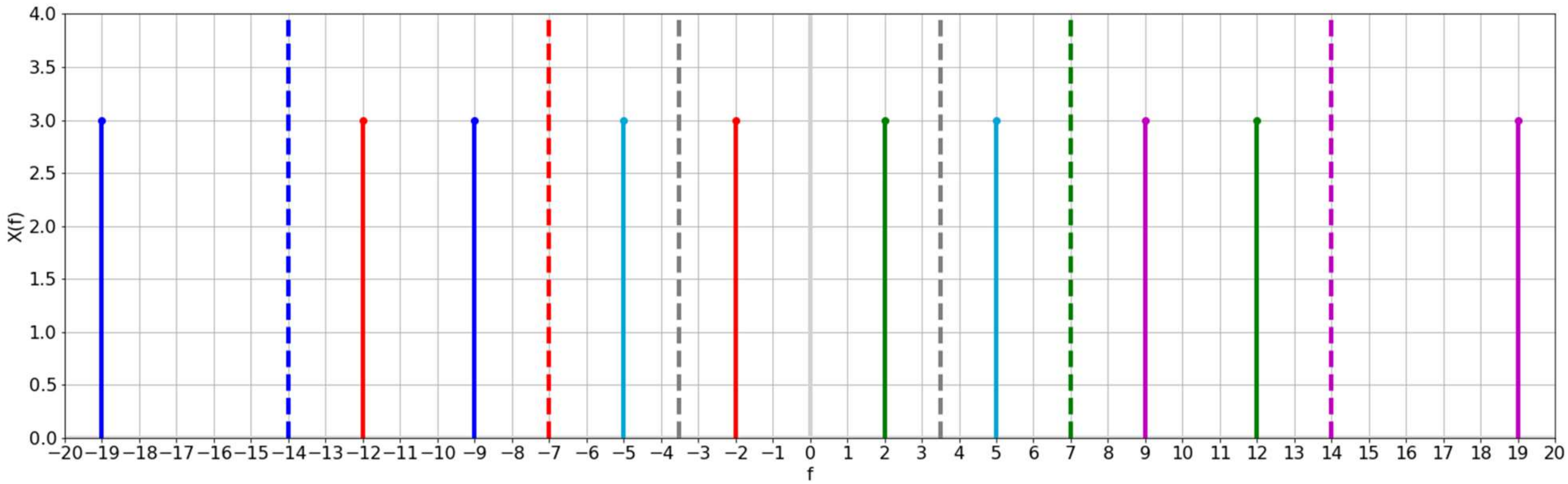

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we consider only domain $-\frac{f_s}{2} < f \leq \frac{f_s}{2}$

Aliasing example – spectrum

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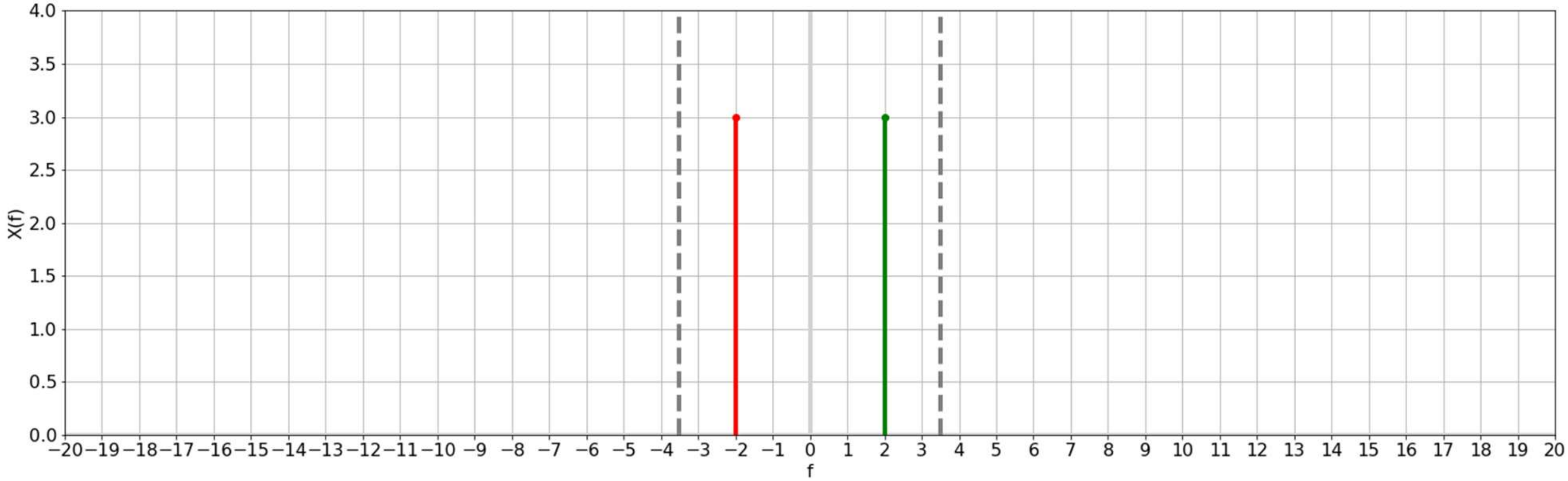


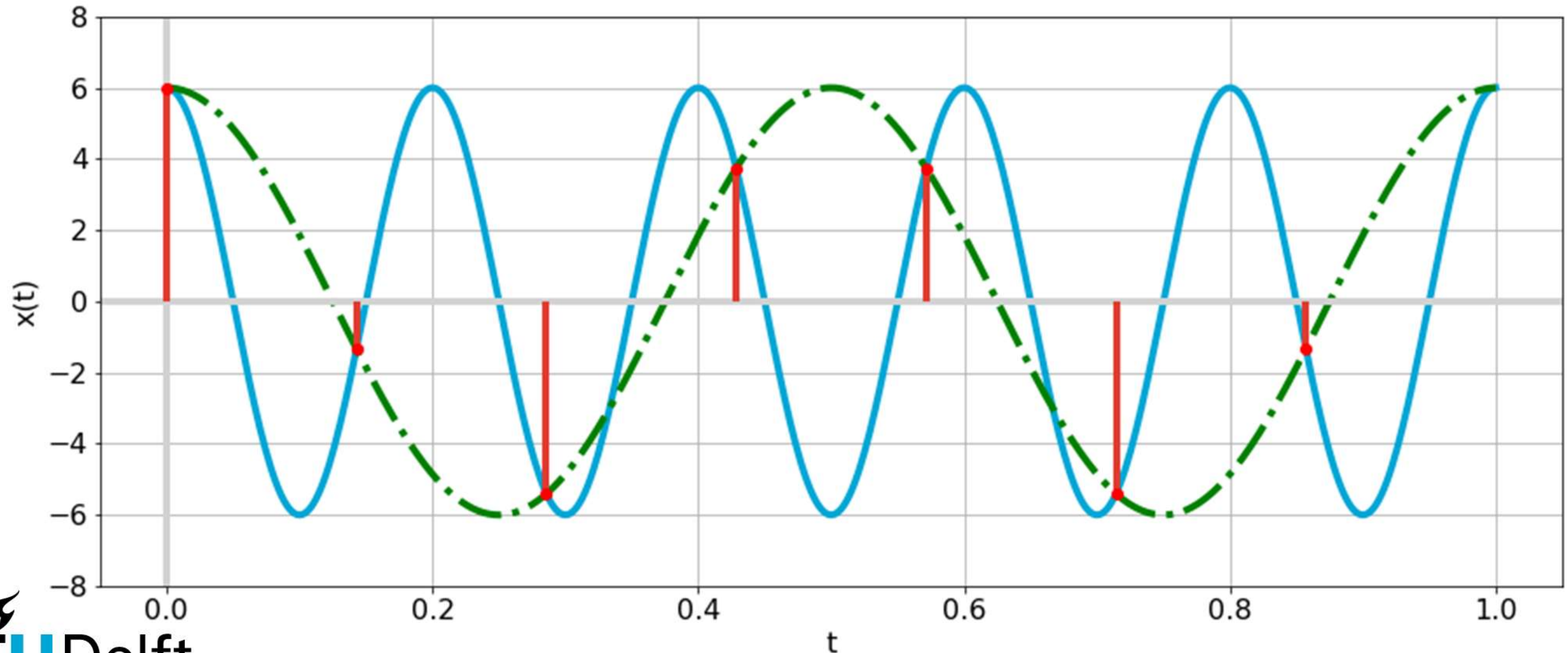
we consider only domain $-\frac{f_s}{2} < f \leq \frac{f_s}{2}$

spectrum of sampled signal will have peaks at -2 and 2 Hz …

Aliasing example – incorrect result

original signal · · sampled & reconstructed signal

$$x(t) = 6\cos(10\pi t)$$



just by eyeballing the samples, a lower frequency signal appears …

Sampling - summary

The Fourier transform of a sampled signal $x_s(t)$ is given as:

$$X_s(f) = \sum_{k=-\infty}^{k=\infty} X(f - kf_s)$$

where f_s is sampling frequency.

To prevent aliases, this frequency f_s should be larger than $2f_h$, where f_h is highest frequency occurring in signal.

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Signal Processing: Discrete Fourier Transform

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Objectives

- Discrete Fourier Transform (DFT)

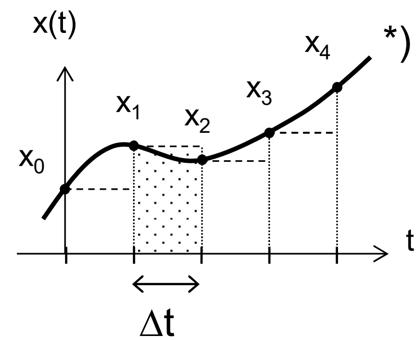
  $\Rightarrow$ tool to be used in practice

  to compute Fourier transform of sampled signal $x_s(t)$

Discrete Time Fourier Transform - DTFT

We discretize Fourier transform of $x_s(t)$ by replacing integral by summation over time:

$$X_s(f) = \sum_{n=-\infty}^{n=\infty} \Delta t \; x_n \; e^{-j2\pi f n\Delta t}$$



with $x_n = x(n\Delta t)$, and where complex exponential is also evaluated at times $t = n\Delta t$, and $n \in \mathbb{Z}$

This is still a continuous function of frequency f ($f \in \mathbb{R}$), periodic with period f_s, exactly as we got with impulse train sampling, and known as Discrete Time Fourier Transform (DTFT).

*) actually function $x(t)\, e^{-j2\pi f}$ rather than just $x(t)$

Discrete Fourier Transform - DFT

We want to analyse spectrum $X_s(f)$ of *sampled* signal $x_s(t)$ using a computer, i.e. by Digital Signal Processing (DSP). Two issues remain, however:

$$X_s(f) = \sum_{n=-\infty}^{n=\infty} \Delta t \, x_n \, e^{-j2\pi f n \Delta t}$$

→ we cannot measure signal forever, so we cannot have $n \rightarrow \infty$.

→ once sampled in time domain, still continuous function (of frequency) does not lend itself to DSP – algorithm requires *discrete* data points as input, and delivers *discrete* data points as output …

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DFT – action item 1

we cannot measure signal forever

sampled signal $x_s(t)$ is sequence x_n with $n = \{-\infty, \ldots, \infty\}$, so values $x_n = x(n\Delta t)$ at discrete times $t = n\Delta t$ (with $n \in \mathbb{Z}$)

we consider it only for time duration $T = N\Delta t$, resulting in just N samples; effectively this means applying window $\Pi\left(\frac{t}{T}\right)$; in practice we set signal to zero outside window, hence: $x_{sw}(t) = \Pi\left(\frac{t}{T}\right) x_s(t)$

This means x_n with $n = 0, \ldots, N-1$ has **finite length**

DFT – action item 2

continuous function does not lend itself to DSP

sample frequency spectrum: we will evaluate it only at discrete frequencies

as we only use piece of $T = N\Delta t$ of signal, *smallest* resulting frequency will be $f_0 = \frac{1}{T} = \frac{1}{N\Delta t} = \frac{f_s}{N}$, known as frequency (or spectral) **resolution**

largest frequency is related to sampling frequency $f_s = \frac{1}{\Delta t}$

hence, spectrum will be computed at frequencies $f = 0, \frac{1}{N} f_s, \frac{2}{N} f_s, \ldots, \frac{N-1}{N} f_s$ (so-called analysis-frequencies)

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DFT

This results in **Discrete Fourier Transform (DFT)**: tool to be used in practice.

DFT turns N samples of signal $x(t)$ into N samples of spectrum $X_{sw}(f)$:

$$x(n\Delta t) \leftrightarrow X_{sw}(kf_0)$$

with both n and $k \in \{0,1,\dots,N-1\}$

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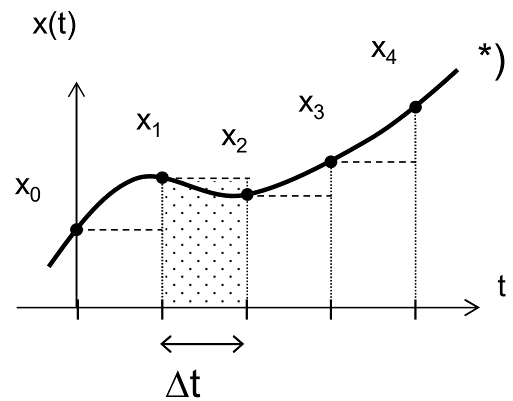
DFT: action-item 1

Sampled, windowed signal $x_{sw}(t)$ equals sequence x_n with $n = 0, \ldots, N-1$.

Then, $X_{sw}(f) = \int_{-\infty}^{\infty} x_{sw}(t) e^{-j2} \quad dt$, which we discretize into:

$$X_{sw}(f) = \sum_{n=0}^{N-1} \Delta t \; x_n \; e^{-j2\pi f n \Delta t}$$

(similar to DTFT)



*) actually function $x(t)\; e^{-j2\pi f}$ rather than just $x(t)$

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DFT: action-item 2

Finally, we sample frequency spectrum, turning $X_{sw}(f)$ into $X_{sws}(f)$ by considering only $f = k\Delta f$ with $\Delta f = f_0 = \frac{1}{T} = \frac{1}{N\Delta t} = \frac{f_s}{N}$ and $k = 0,1,\dots,N-1$:

$$X_{sws}(k\Delta f) = \Delta t \sum_{n=0}^{N-1} x_n e^{-j2\pi k\Delta f n\Delta t} = \Delta t \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn} \qquad \text{with } k \in \mathbb{Z}$$

Hence sequence X_k equals $X_{sws}(f)$ at $f = k\Delta f$ for $k = 0,1,\dots,N-1$.

$$X_k = \Delta t \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn}$$

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DFT - summary

$$X_k = \Delta t \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn}$$

$$x_n = \frac{1}{N\Delta t} \sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi}{N}kn}$$

with both k and $n \in \{0,1,\dots,N-1\}$

With X_k, we consider function $X(k\Delta f)$ by restoring *frequency dimension*,

frequency resolution: $\Delta f = f_0 = \frac{1}{T} = \frac{1}{N\Delta t} = \frac{f_s}{N}$

With x_n, we consider function $x(n\Delta t)$ by restoring *time dimension*,

time resolution: $\Delta t = \frac{1}{f_s}$

DFT - summary

In many textbooks, we find DFT as:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn}$$

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi}{N}kn}$$

with both k and $n \in \{0,1,\ldots,N-1\}$

Hence, without factors Δt and $\frac{1}{\Delta t}$. This is also how DFT is implemented in programming languages like Matlab and Python; user has to restore time and frequency dimension!

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Signal Processing: Spectral Estimation

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Objectives

- energy and power signals
- Parseval's theorem
- Power Spectral Density (PSD)
- basic spectral estimation: periodogram

so far looked at amplitude spectrum resulting from Fourier Series and magnitude spectrum from Fourier transform, $|X_k|$ and $|X(f)|$ resp; can we connect amplitudes at different frequencies to physical notions of energy and power?

how is energy/power of signal distributed over frequency → spectral analysis

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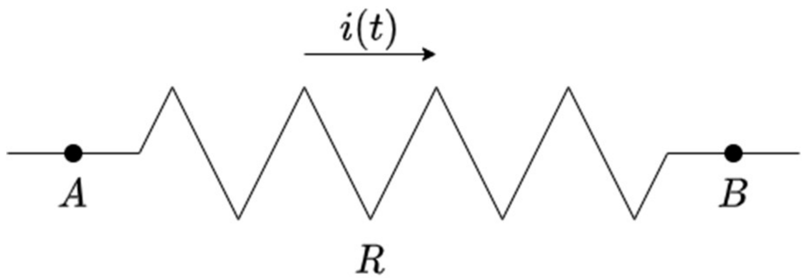
Energy and power signals

Definitions of energy and power stem from electrical engineering. Suppose $u(t)$ is voltage across resistor R producing current $i(t)$



Energy and Power signals

Instantaneous power is defined as $p(t) = u(t)i(t)$, and $u(t) = i(t)R$, so

$p(t) = i^2(t)R = \frac{u^2(t)}{R}$ $\qquad$ $u(t)$ in [V], $i(t)$ in [A]

with $R = 1\ \Omega$, instantaneous power per Ohm is given as:

$$p(t) = u(t)i(t) = i^2(t) = u^2(t)$$

Integrating over $|t| \leq T$, we define total energy and average power as:

(on a per Ohm basis)

$E = \lim_{T\to\infty} \int_{-T}^{T} u^2(t)dt$ $\qquad$ in Joule [J]

$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u^2(t)dt$ $\qquad$ in Watt [W]

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Energy and power signals

For signal $x(t)$, total energy (normalized to unit resistance) is defined as:

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

in Joule [J = V A s = N m]

and average power (normalized to unit resistance) as:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

in Watt [W = J/s]

For real signals, modulus signs may be removed from equations above.

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Parseval's theorem (Fourier transform)

We obtain Parseval's theorem for Fourier transforms:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \underbrace{|X(f)|^2}_{\text{Energy Spectral Density (ESD)}} df$$

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Parseval's theorem (DFT)

Note that X_k denotes DFT-coefficients, with Δt included $\longrightarrow$ $X_k = \Delta t \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn}$

Power of signal, contained in frequency band of width $\Delta f = \frac{1}{T}$, at frequency $f = k\,\Delta f$ is:

$\boxed{S(k\Delta f) = \frac{1}{T}|X_k|^2}$ in [W/Hz] for $k = 0, \dots N-1$

and actually is power **density**

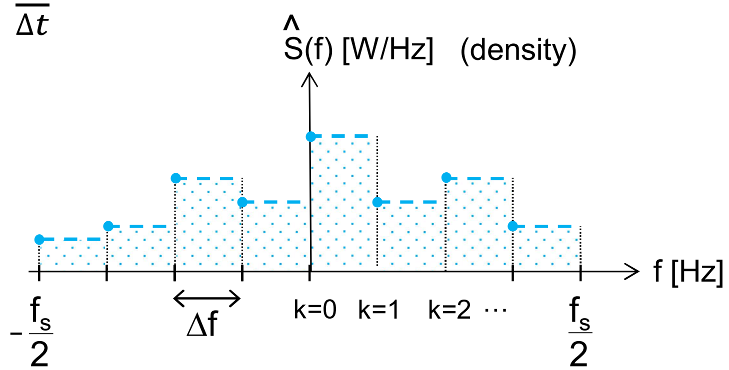
Parseval's theorem (DFT) → periodogram

$\boxed{S(k\Delta f) = \frac{1}{T}|X_k|^2}$ in [W/Hz] for $k = 0, \dots N-1$ and $\Delta f = \frac{1}{T}$

This turns out to be an *estimate for* the PSD, and is referred to as periodogram (estimate may be indicated by hat-symbol, hence $\hat{S}$).

Product $\Delta f\ S(k\Delta f)$ is contribution by frequency *band* with width Δf at frequency $f = k\Delta f$, to power P of signal.

Periodogram $S(f)$ defined for $0 \leq f < f_s$, or equivalently $-\frac{f_s}{2} < f \leq \frac{f_s}{2}$ (two-sided), with $f_s = \frac{1}{\Delta t}$



Note that X_k denotes DFT-coefficients, with Δt included:

$$X_k = \Delta t \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn}$$