Modelling, Uncertainty and Data for Engineers (MUDE)

Signal Processing: continuous time
Fourier series & Fourier transform

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Modelling, Uncertainty and Data for Engineers (MUDE)

Signal Processing: real Fourier Series

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Objectives

goal is to express periodic signal $x(t)$ as a sum of harmonically related cosines and sines:

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{k=\infty} b_k \sin(k\omega_0 t)$$

- periodic functions (re-cap)
- Fourier Series (real, trigonometric)

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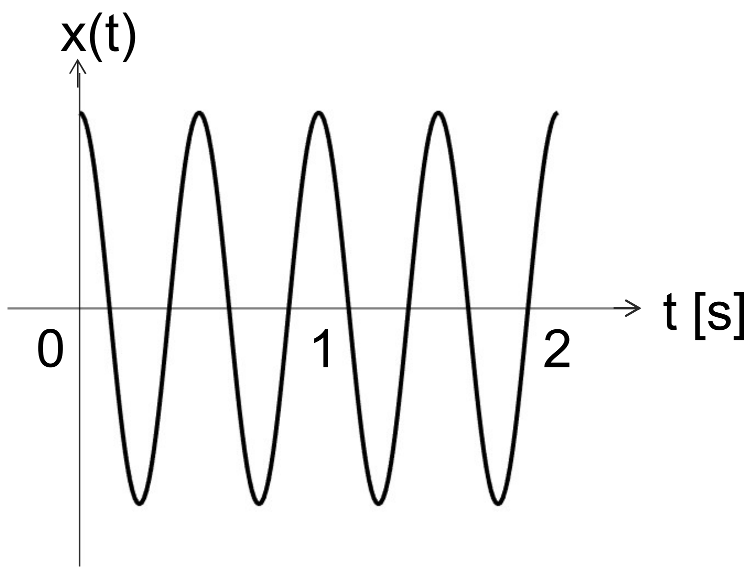
Periodic functions (re-cap)

Example: $x(t) = A\cos(2\pi f_0 t + \theta)$

- amplitude A [e.g. V, m, m/s^2]
- frequency f_0 [Hz]

    angular frequency $\omega_0 = 2\pi f_0$ [rad/s]

    period $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$ [s]
- phase θ [rad]



$f_0 = 2 \qquad \theta = 0$

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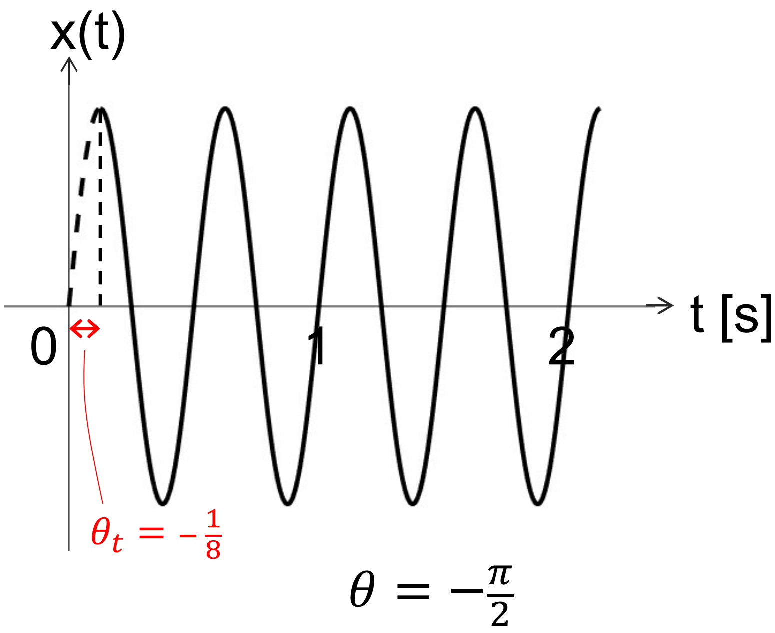
Periodic functions (re-cap)

Example: $x(t) = A\cos(2\pi f_0 t + \theta)$

- amplitude A [e.g. V, m, m/s^2]
- frequency f_0 [Hz]
  angular frequency $\omega_0 = 2\pi f_0$ [rad/s]
  period $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$ [s]
- phase θ [rad]
- time-delay $\theta_t = \frac{\theta}{2\pi f_0}$ [s]
  as $x(t) = A\cos(2\pi f_0 (t + \theta_t))$



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Periodic functions (re-cap)

A deterministic function/signal is **periodic** if:

$$x(t + T_0) = x(t) \qquad \text{for } -\infty < t < \infty$$

where T_0 is **period** of signal.

Smallest value of T_0 which satisfies equation above is **fundamental period.**

If equation is not satisfied for any value of T_0, signal is **aperiodic.**

Periodic functions (re-cap)

We consider signals in single dimension, typically time t (independent variable), hence $x(t)$.

Instead we can have $x(r)$, with r (1D) position coordinate, or even multi-variate position vector $\boldsymbol{r}$.

default: time *t* [s], frequency *f* [Hz]

and today $t \in \mathbb{R}$ (continuous time)

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Trigonometric series

Apparently there exists possibility of building up arbitrary periodic signal from *sums of harmonically related sinusoidal terms*. So:

> Given specific periodic signal, how do we find its trigonometric series representation?

It will be shown that resulting series, *unique* for each periodic signal, is called **trigonometric Fourier series** of that signal, sometimes also real Fourier series, or trigonometric polynomial, i.e. sum of cosines and sines, all with zero phase.

Fourier Series

The general form of real trigonometric Fourier Series:

$$x(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \cdots + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \cdots$$

More compact:

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{k=\infty} b_k \sin(k\omega_0 t) \quad \text{with } \omega_0 = \frac{2\pi}{T_0}$$

with integer values for k. Note that as right-hand side is sum of *harmonically-related* sinusoids, so must left-hand side (i.e. be periodic).

Now, problem is *how to find coefficients* a_0*,* a_k *and* b_k *such that* $x(t)$ *is represented (or approximated) best.*

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Fourier Series – summary

Any periodic signal*) can be written into series of harmonically related sinusoids:

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{k=\infty} b_k \sin(k\omega_0 t) \qquad \text{with } \omega_0 = \frac{2\pi}{T_0}$$

where coefficients can be found as:

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)dt \qquad b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)dt \qquad \text{with } k \in \mathbb{N}^+$$

*) signal $x(t)$ is assumed to be absolutely integrable over one period, hence $\int_{T_0} |x(t)|dt < \infty$
(and there is another condition)

Fourier Series – conclusion

Can we express basically any periodic function/signal as a sum of just cosines and sines with different frequencies?

Yes, we can:

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{k=\infty} b_k \sin(k\omega_0 t)$$

with $\omega_0 = \frac{2\pi}{T_0}$

with coefficients a_0, a_k and b_k as on previous slide.

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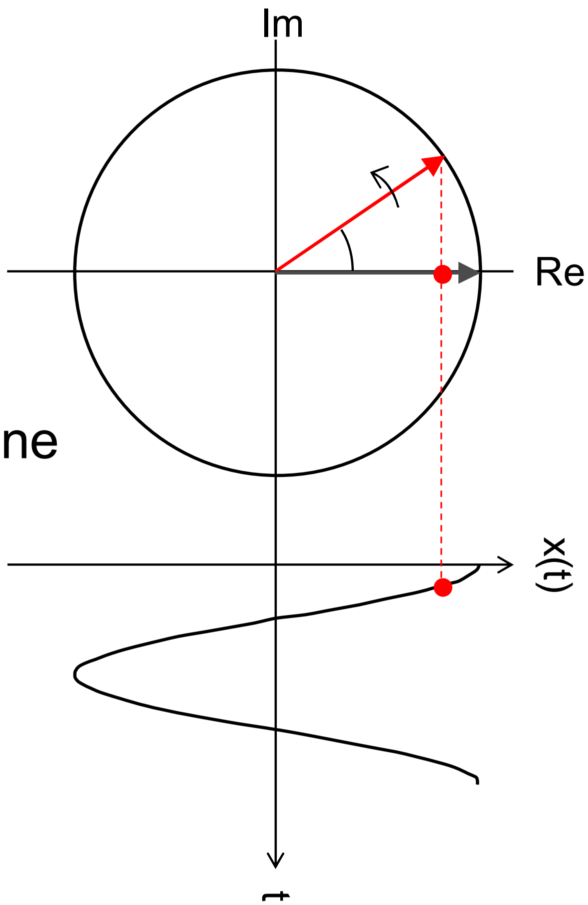
Periodic functions (re-cap)

Cosine and sine can be thought of as originating from an object travelling over a (unit) circle; the object is rotating at angular frequency ω_0



The circle is interpreted as being in the complex plane

$$\tilde{x}(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

position of object is described by complex number (vector) $\tilde{x}(t)$

● = $Re\left(\tilde{x}(t)\right) = \cos(\omega_0 t)$

$Im\left(\tilde{x}(t)\right) = \sin(\omega_0 t)$

for $A = 1$ and $\theta = 0$

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Signal Processing: complex Fourier Series

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Re-cap: trigonometric real Fourier Series

Now, same signal decomposition, but in different form, using *complex algebra,* as commonly done in spectral analysis of signals

We're on our way to analyse signal in terms of *frequency*, rather than in *time* domain

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Re-cap: complex algebra

j: imaginary unit (or number), $j^2 = -1$

e: Euler's number (base of natural logarithm) e = 2.71…

Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

and therewith:

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta) \quad \rightarrow \quad \cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$e^{j\theta} - e^{-j\theta} = 2j\sin(\theta) \quad \rightarrow \quad \sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

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Complex Fourier Series (derivation)

substitute complex exponential forms of $\cos(k\omega_0 t)$ and $\sin(k\omega_0 t)$ into trigonometric series:

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} a_k \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + \sum_{k=1}^{k=\infty} b_k \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} \underbrace{\frac{1}{2}(a_k - jb_k)}_{X_k} e^{jk\omega_0 t} + \sum_{k=1}^{k=\infty} \underbrace{\frac{1}{2}(a_k + jb_k)}_{X_{-k}} e^{-jk\omega_0 t}$$

Complex Fourier Series (derivation)

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} X_k e^{jk\omega_0 t} + \sum_{k=1}^{k=\infty} X_{-k} e^{-jk\omega_0 t}$$

in this summation replace k by $-k$

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} X_k e^{jk\omega_0 t} + \sum_{k=-1}^{k=-\infty} X_k e^{jk\omega_0 t}$$

$X_0 = a_0$ and $e^{j0\omega_0 t} = 1$ k=0

(real number, equals signal average)

$$x(t) = \sum_{k=-\infty}^{k=\infty} X_k e^{jk\omega_0 t}$$

with coefficients $k \in \mathbb{Z}$

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} \, dt$$

the complex exponential Fourier series (double sided)

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Complex Fourier Series

$$x(t) = \sum_{k=-\infty}^{k=\infty} X_k e^{jk\omega_0 t}$$

First equation is known as **synthesis** equation of Fourier Series, as it constructs (‘synthesizes’) signal using complex exponential basis functions.

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt \qquad k \in \mathbb{Z}$$

Second equation is known as **analysis** equation of Fourier Series, as it allows us to analyse how signal can be represented by complex exponential basis functions (where index k refers to frequency $k\omega_0$).

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Line spectra

complex exponential Fourier Series coefficients

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt \qquad\qquad k \in \mathbb{Z}$$

show amplitudes of all phasors involved, through modulus of complex exponential coefficients $|X_k|$ versus frequency kf_0, yields **amplitude spectrum**

show phases of all phasors, through argument of complex exponential coefficients θ_k versus frequency kf_0, yields **phase spectrum**

Note: $\omega_0 = 2\pi f_0$

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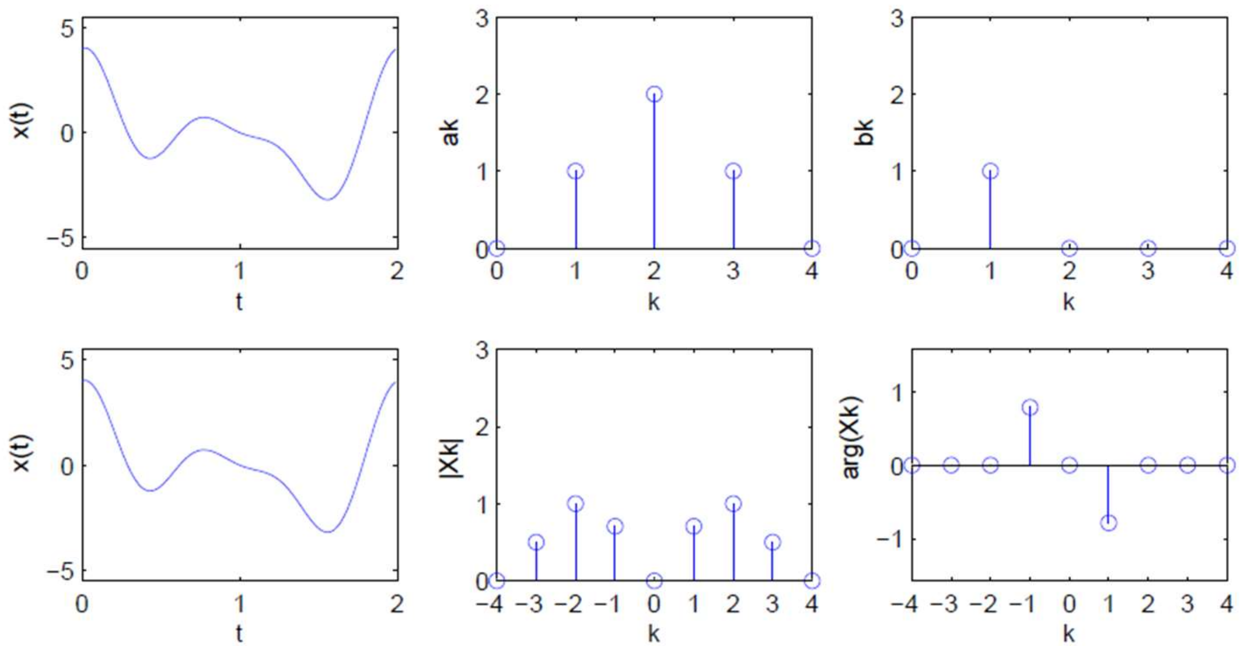
Line spectra - example

Consider following signal, with period $T_0 = 2$ s, so $f_0 = 0.5$ Hz, composed of three cosines and one sine:

$$x(t) = 1\cos(2\pi f_0 t) + 2\cos(2\pi 2 f_0 t) + 1\cos(2\pi 3 f_0 t) + 1\sin(2\pi f_0 t)$$

real and complex Fourier coefficients are shown below:



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Signal Processing: Fourier transform

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Objectives

Can a-periodic signals ($T_0 \rightarrow \infty$) also be written as 'sum' of cosine and sine functions?

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Re-cap: complex Fourier Series (for periodic signal)

complex exponential Fourier Series:

$$x(t) = \sum_{k=-\infty}^{k=\infty} X_k e^{jk2\pi f_0 t}$$

with $\omega_0 = 2\pi f_0$ and $k \in \mathbb{Z}$

complex coefficients can be found as:

$$X_k = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jk2\pi \ {}_0 t} dt$$

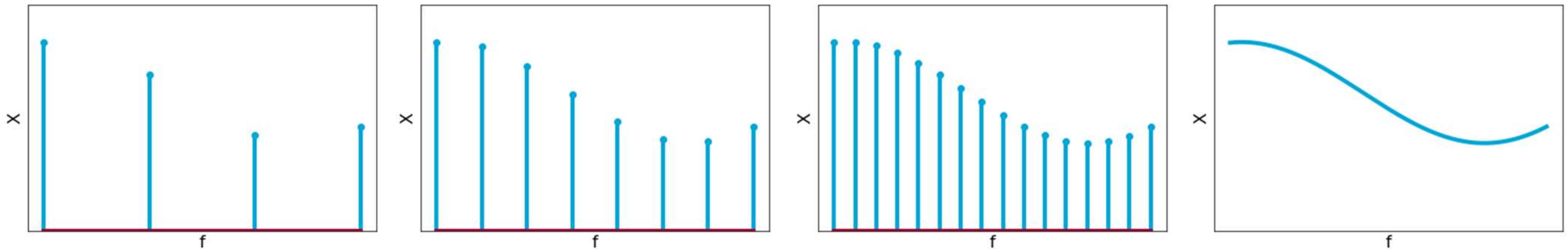
the integration over <u>one period</u>, $\int_{T_0}$, is conveniently chosen here symmetric about zero

Fourier transform - introduction

When period T_0 increases, frequencies belonging to Fourier Series coefficients (kf_0) lie closer and closer together, because $f_0 = \frac{1}{T_0}$ decreases.

Now, what if T_0 approaches infinity? That is, what would happen if we have an a-periodic signal?

Fourier coefficients will lie infinitesimally close to each other, so they define continuous function of frequency $f \approx kf_0$



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Fourier transform - derivation

Now, integral within parentheses is defined as **Fourier integral** or (continuous-time) **Fourier transform** $\mathcal{F}()$:

$$X(f) = \int_{t=-\infty}^{t=\infty} x(t)e^{-j2\pi f}\ dt$$

Complex number X_k with Fourier Series now got complex function of frequency f: $X(f)$

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Fourier transform - derivation

Inverse Fourier transform $\mathcal{F}^{-1}()$ can also be derived:

$$x(t) = \int_{f=-\infty}^{f=\infty} X(f)\, e^{j2\pi f t} df$$

Note: to obtain $x(t)$ from $X(f)$, we integrate over *frequency* f, the result is function of *time* t.

Hence, Fourier transform and its inverse can be used to transform time signal to frequency domain, and other way around. This yields a Fourier transform pair:

$$x(t) \xrightarrow{\mathcal{F}} X(f) \xrightarrow{\mathcal{F}^{-1}} x(t)$$ or $$x(t) \leftrightarrow X(f)$$

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Fourier transform – amplitude and phase

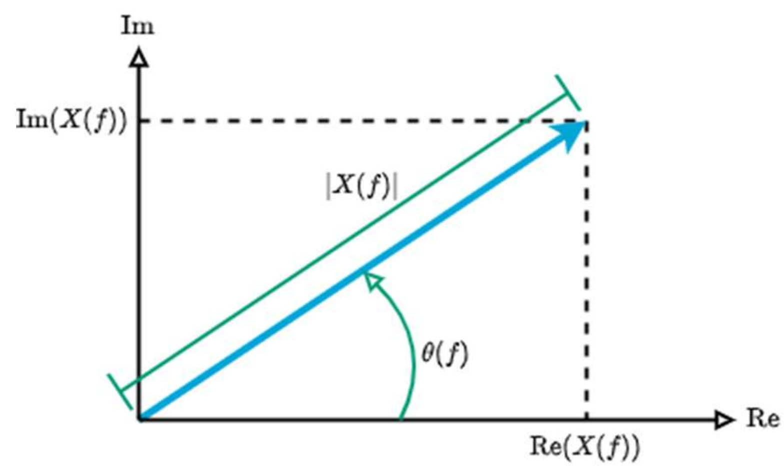
Like coefficients of complex Fourier Series, Fourier transform can be written in terms of magnitude and phase:

$$X(f) = |X(f)|e^{j\theta(f)}$$

with:

$$|X(f)| = \sqrt{\left(Re\left(X(f)\right)\right)^2 + \left(Im\left(X(f)\right)\right)^2}$$

$$\theta(f) = \arctan\left(\frac{Im\left(X(f)\right)}{Re\left(X(f)\right)}\right)$$



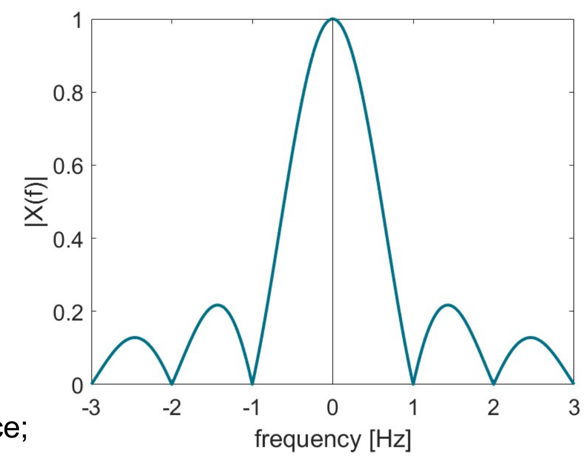
Fourier transform – amplitude and phase

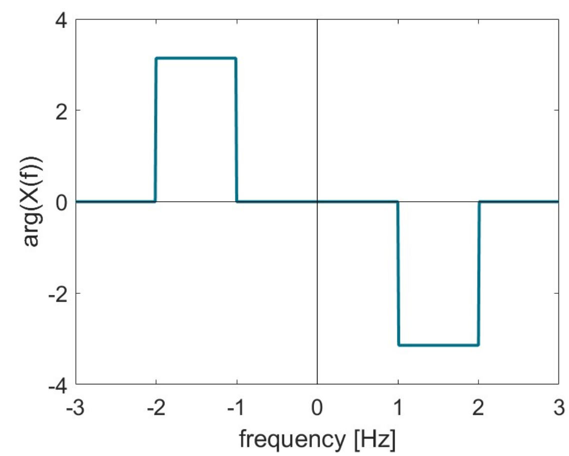
When $x(t)$ is real, then:

$|X(f)| = |X(-f)|$ and $\theta(f) = -\theta(-f)$

Magnitude $|X(f)|$ is *even* function of f → **amplitude spectrum**[*)]

Phase $\theta(f)$ is *odd* function of f → **phase spectrum**





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*) the term amplitude spectrum is commonly used in practice; formally it should read magnitude spectrum