The finite element method

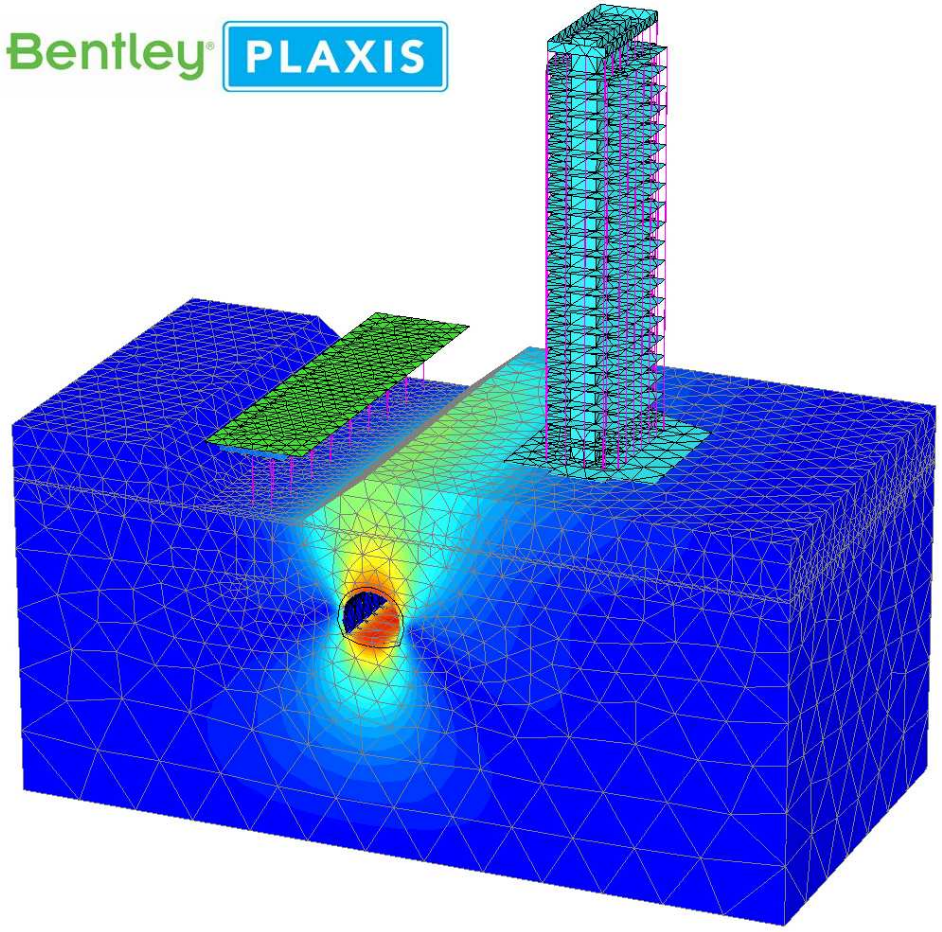
MUDE week 2.2

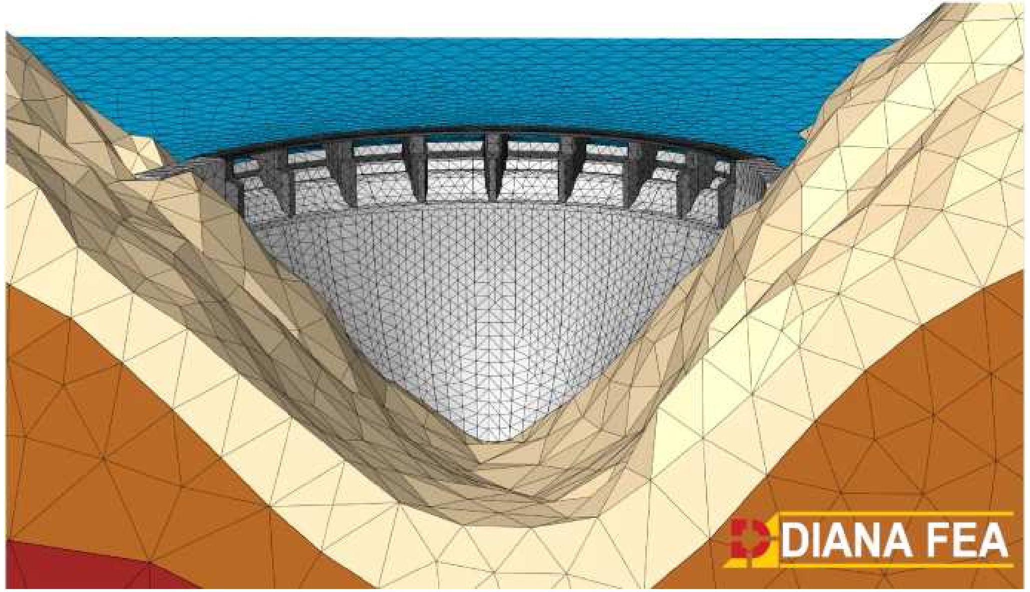
Frans van der Meer

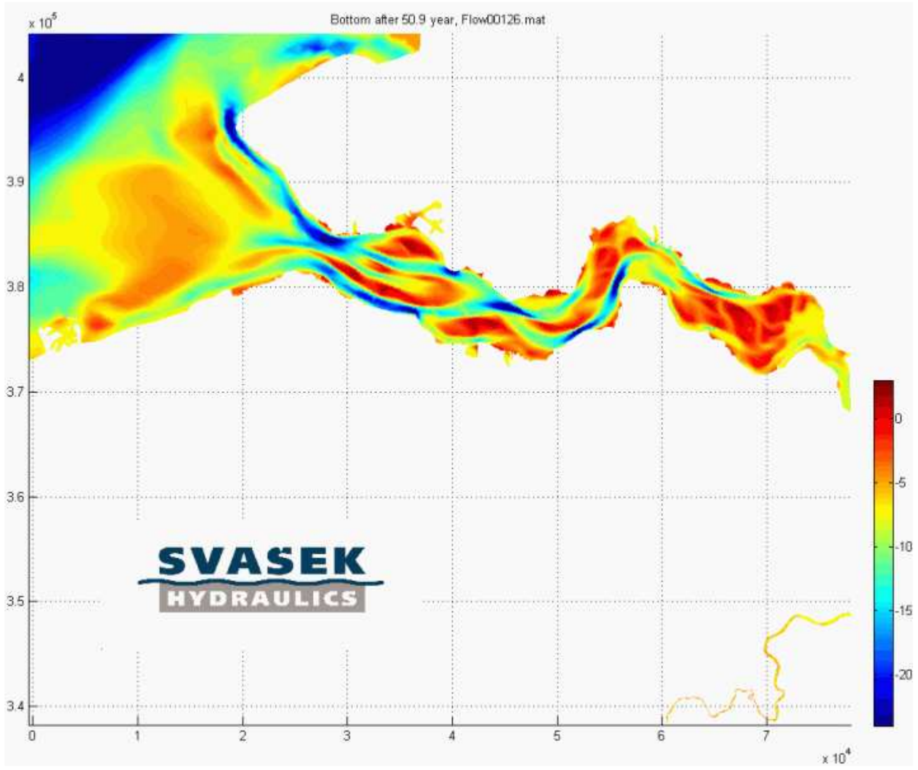
TU Delft

Finite elements and CEG

Three commercial codes with strong ties to this faculty







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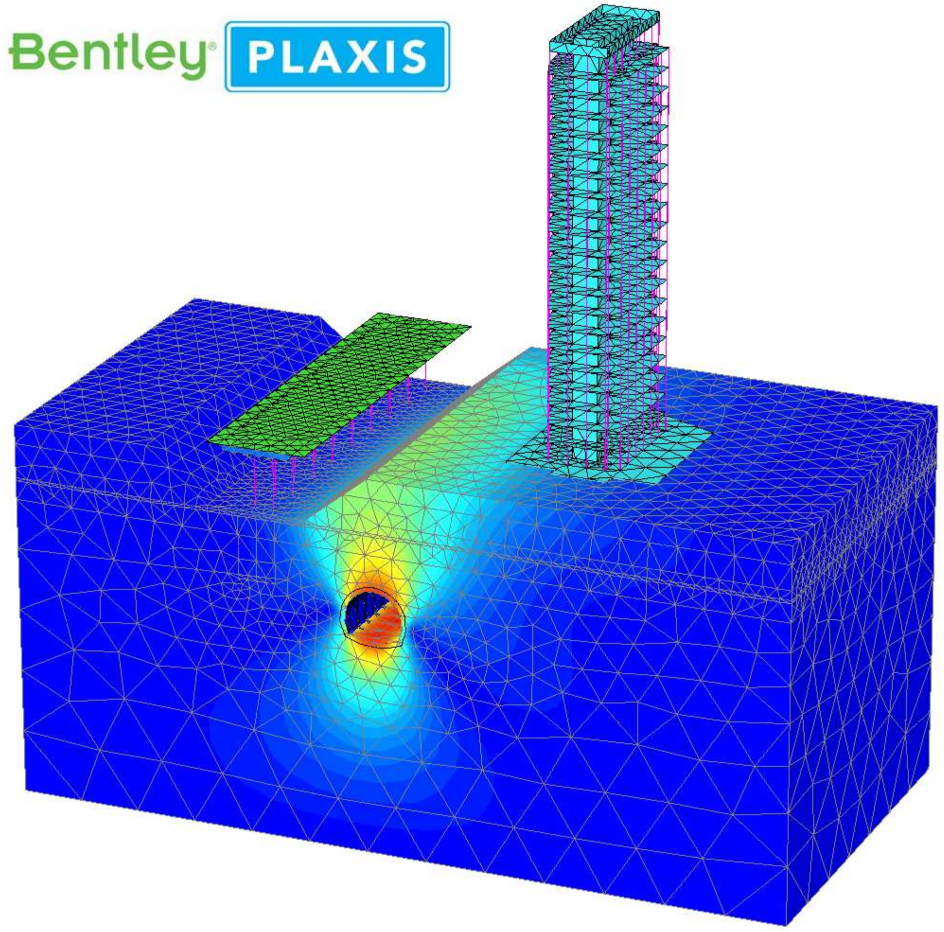

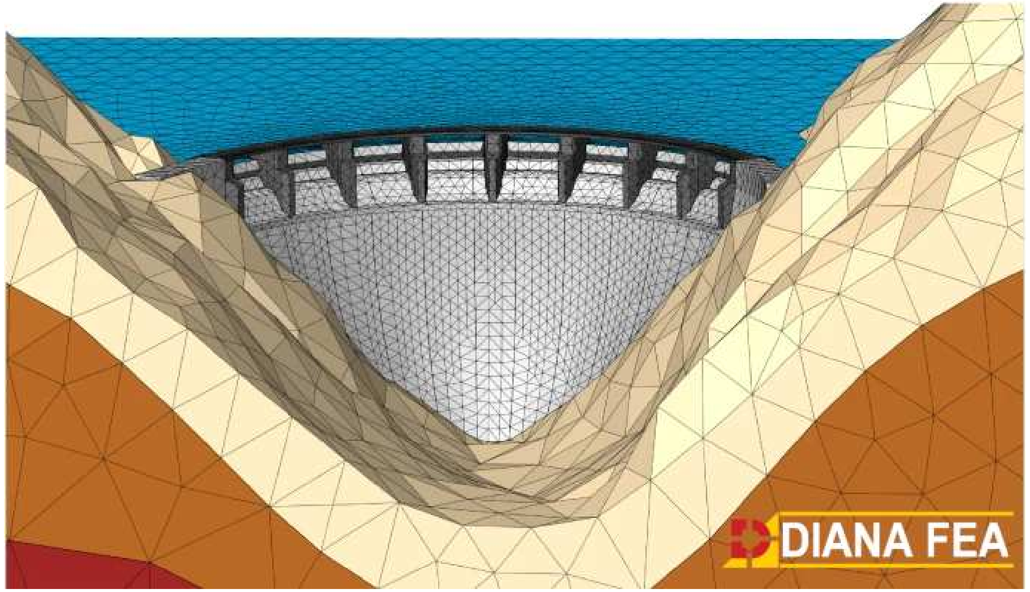

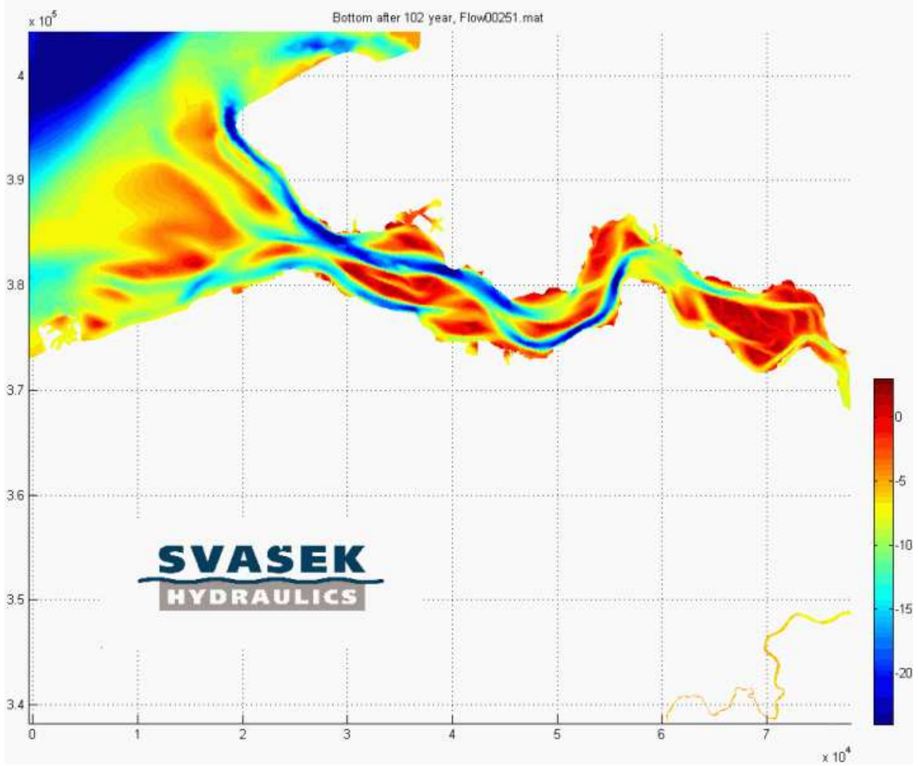

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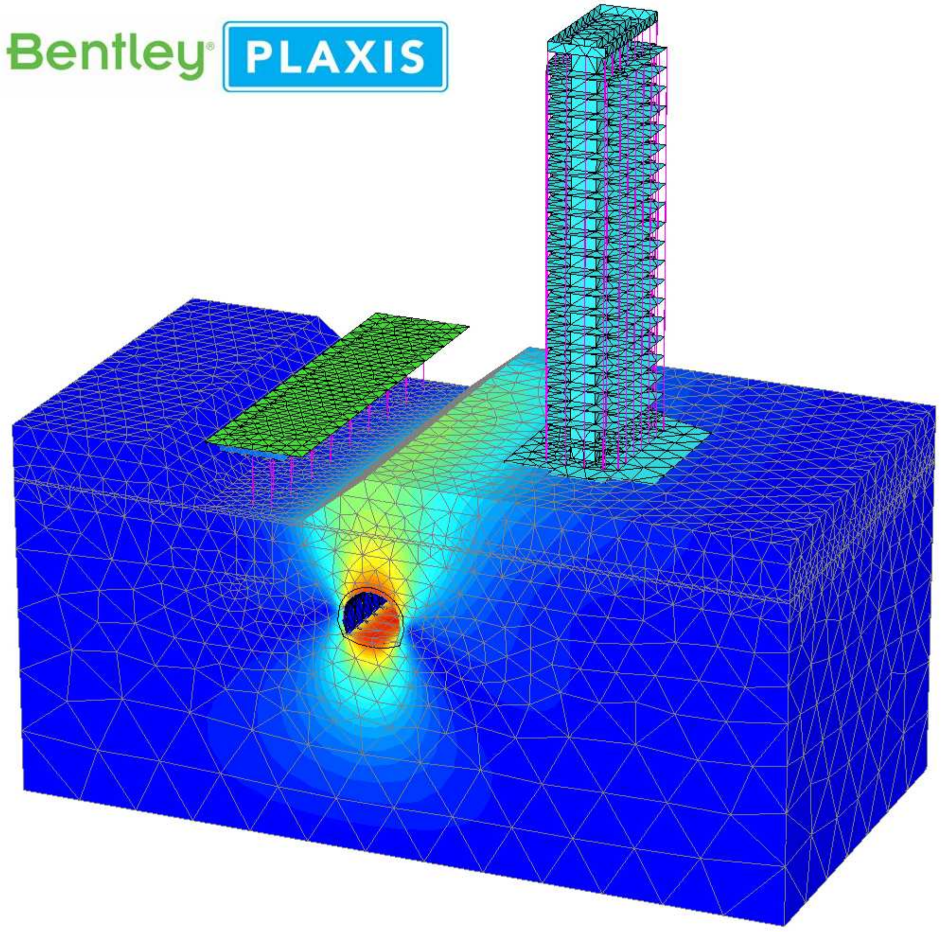

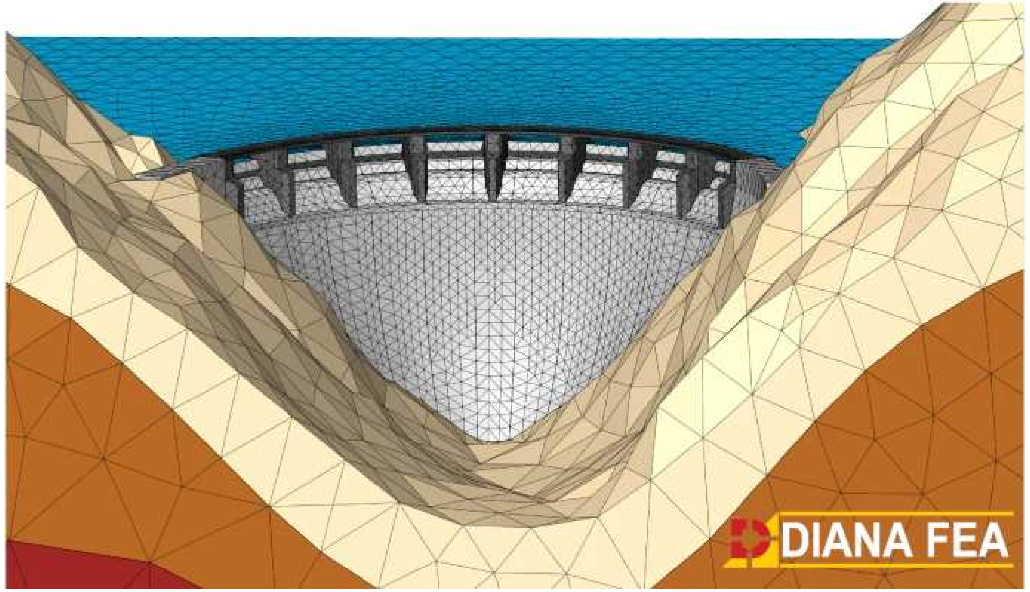

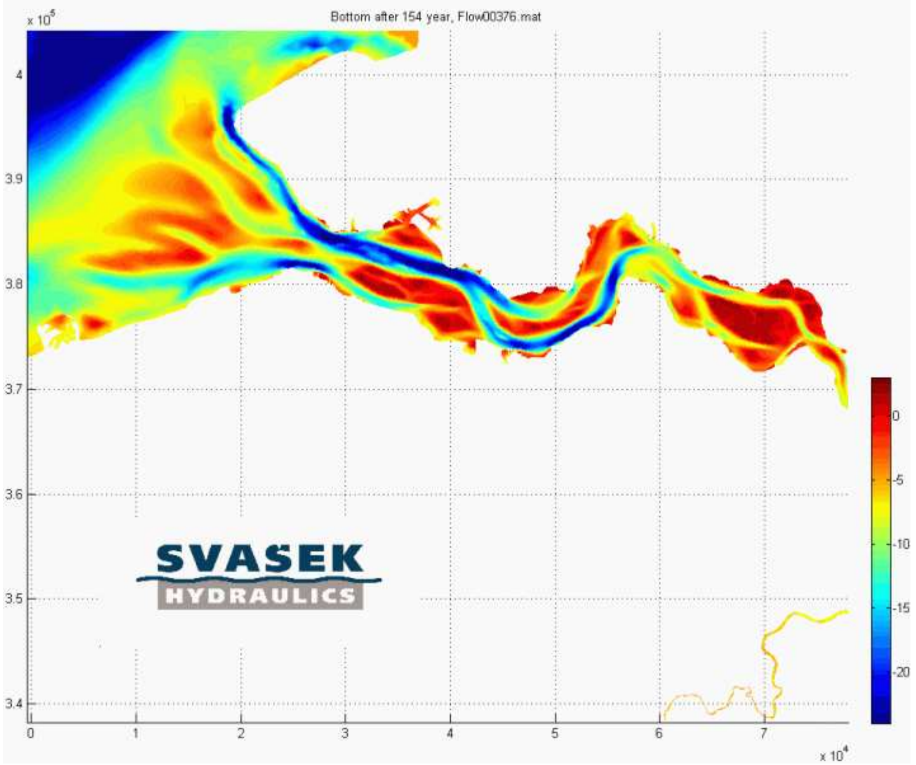

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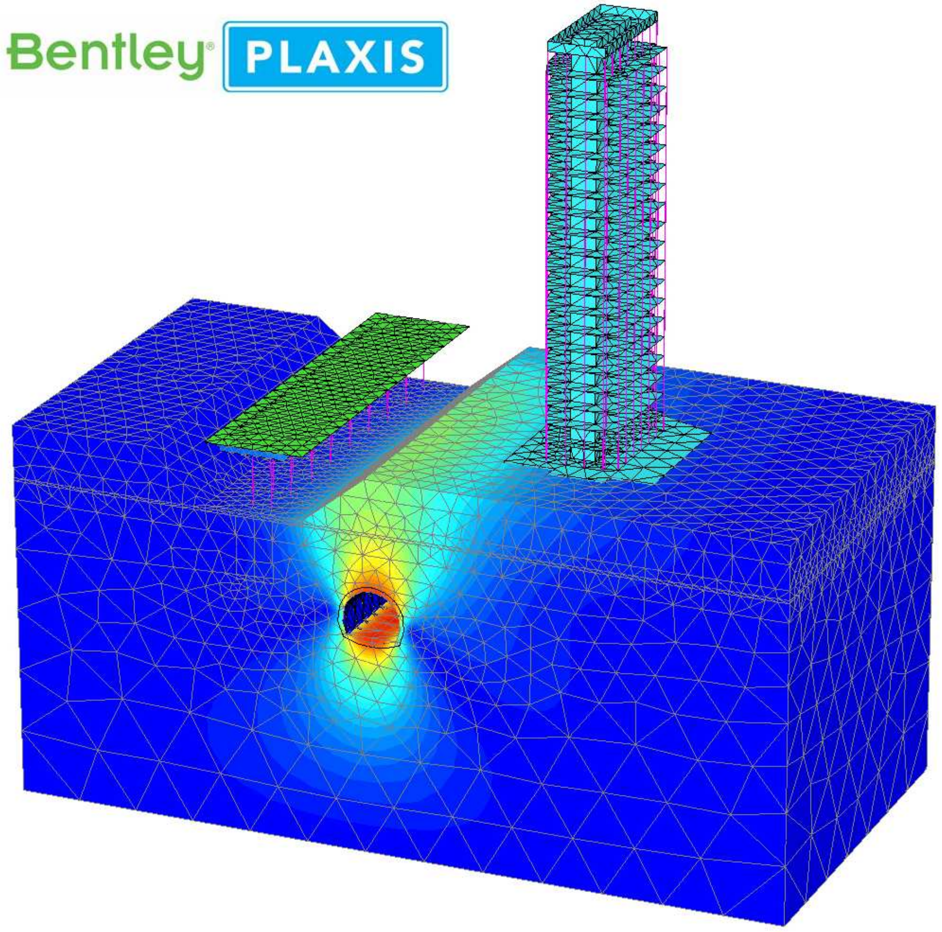

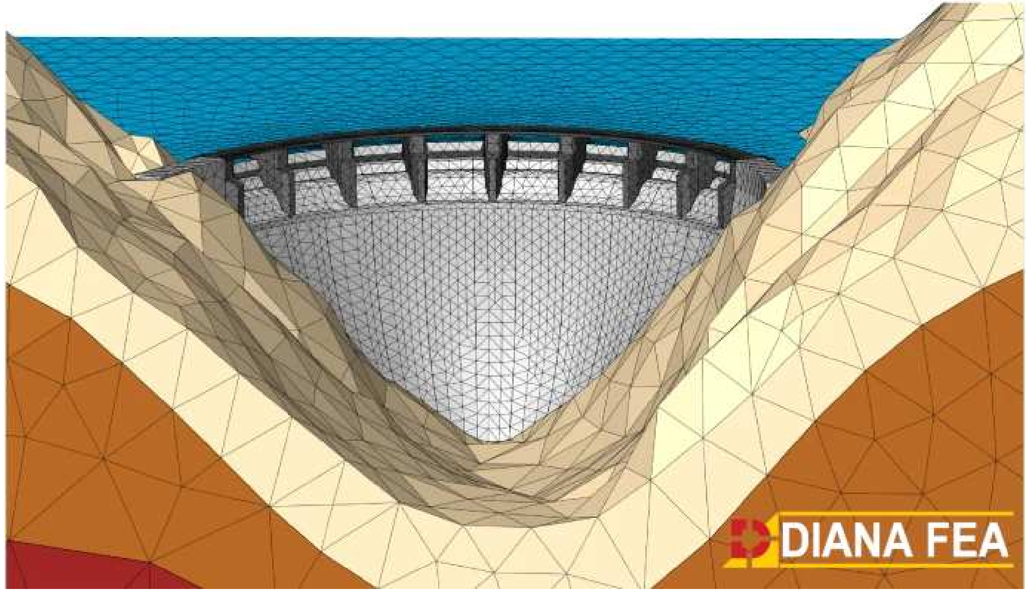

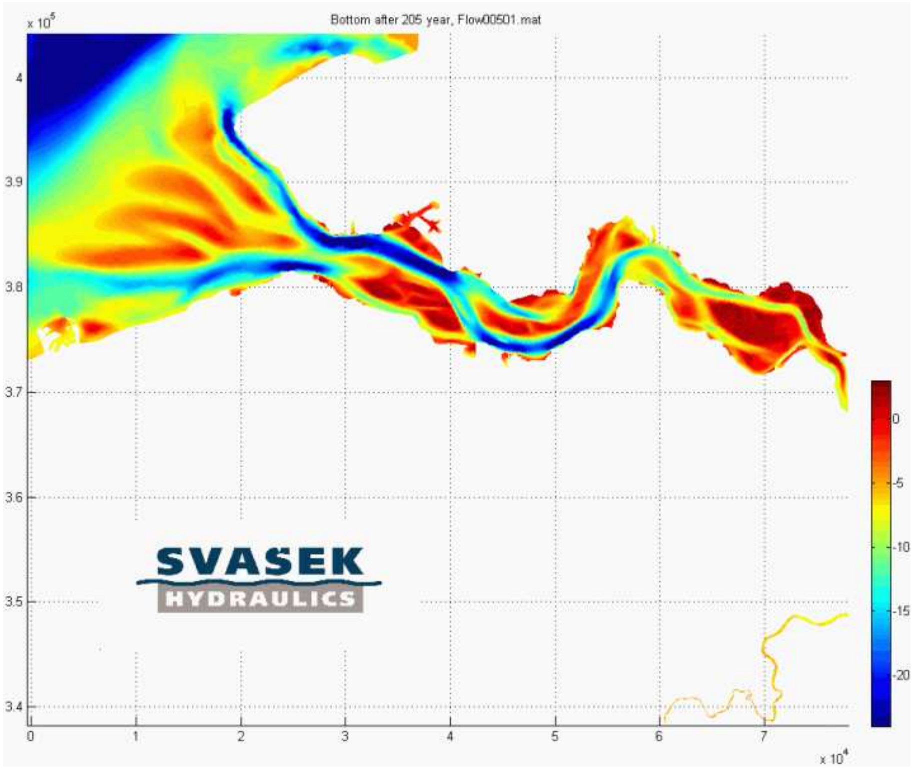

Finite elements and CEG

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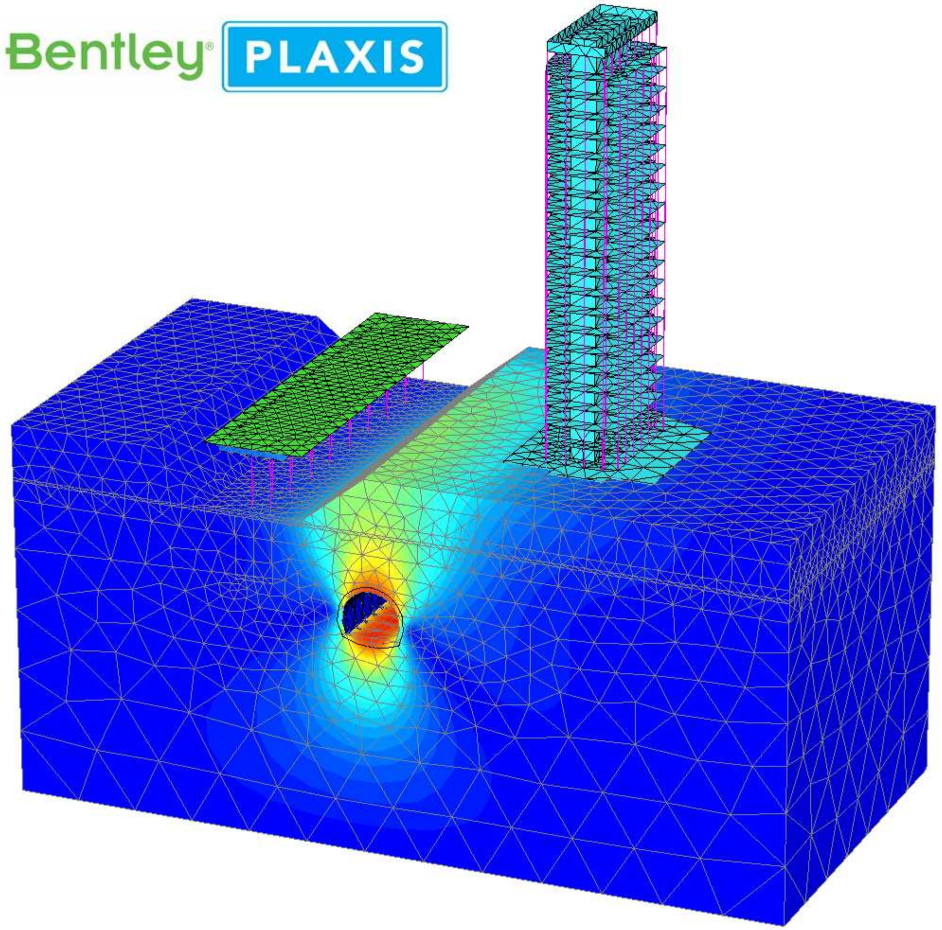
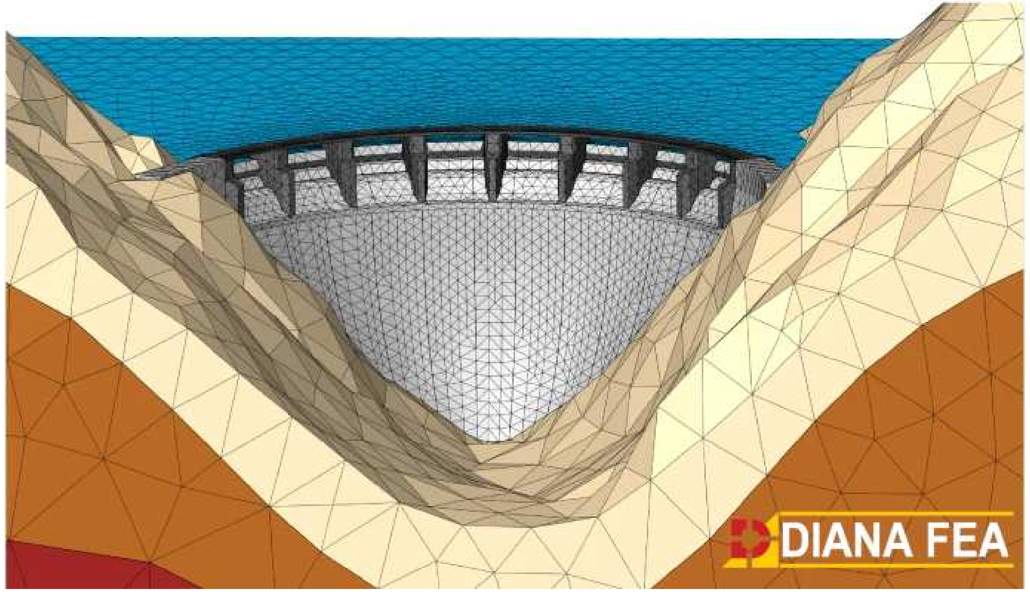
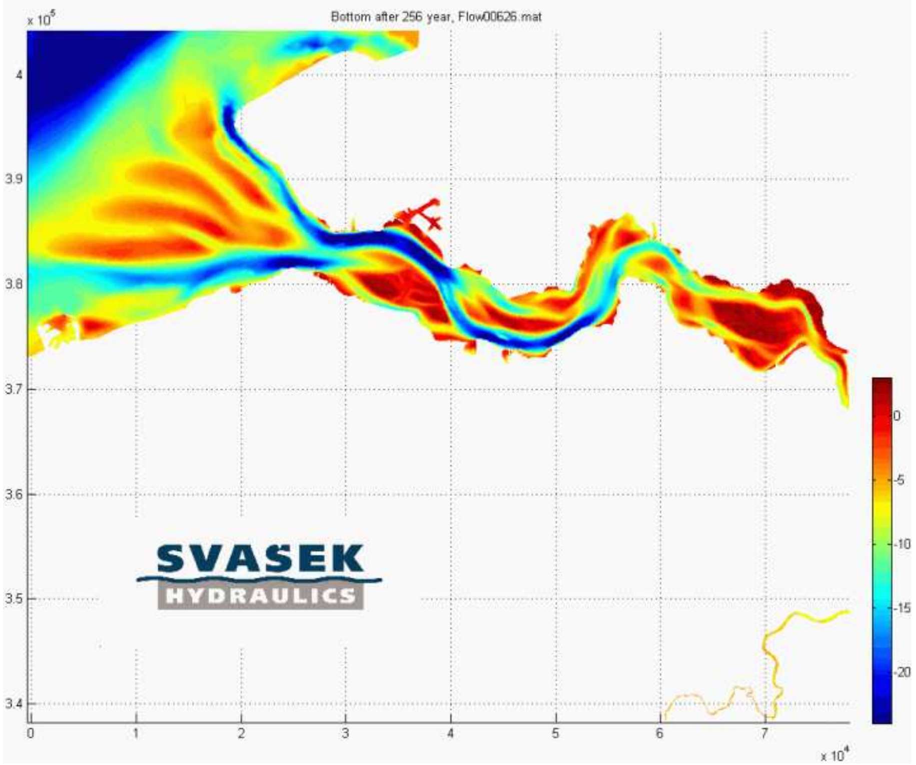
Bentley PLAXIS







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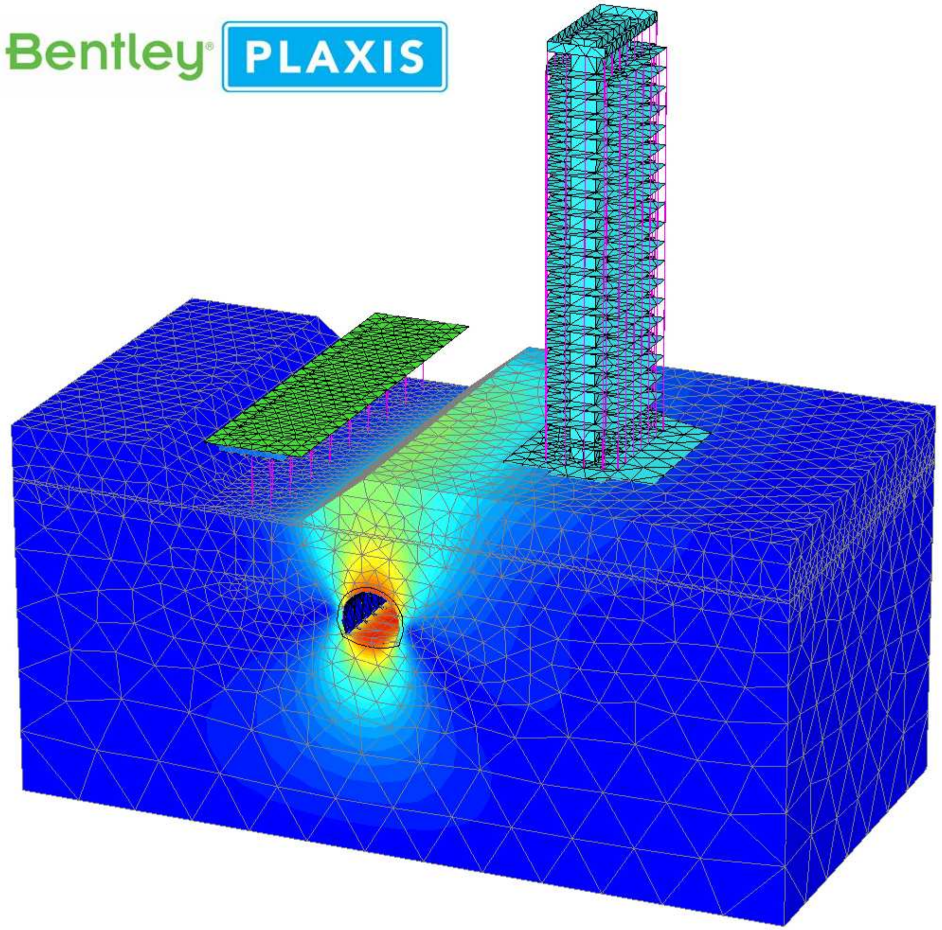

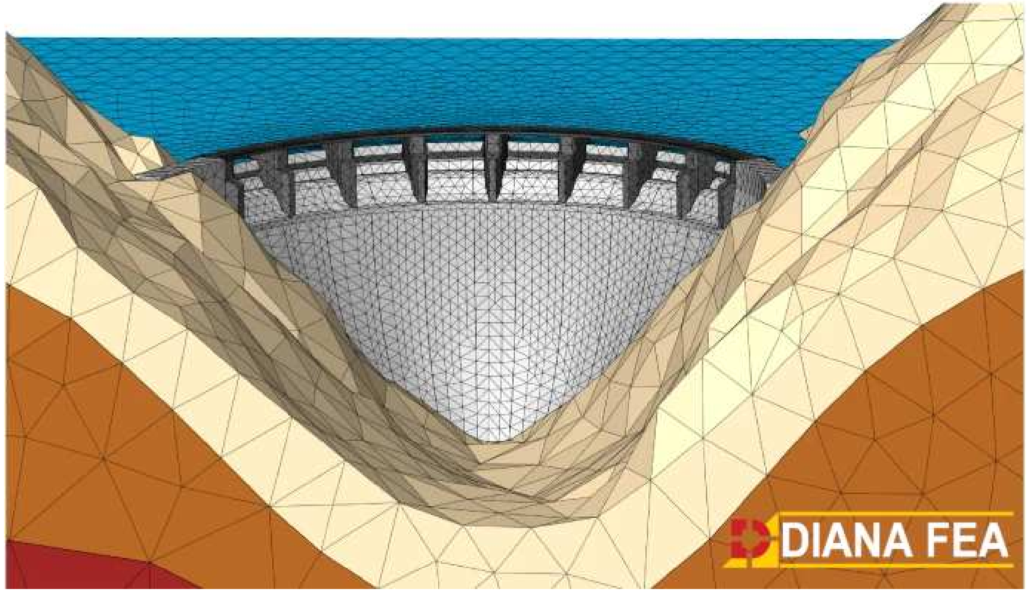

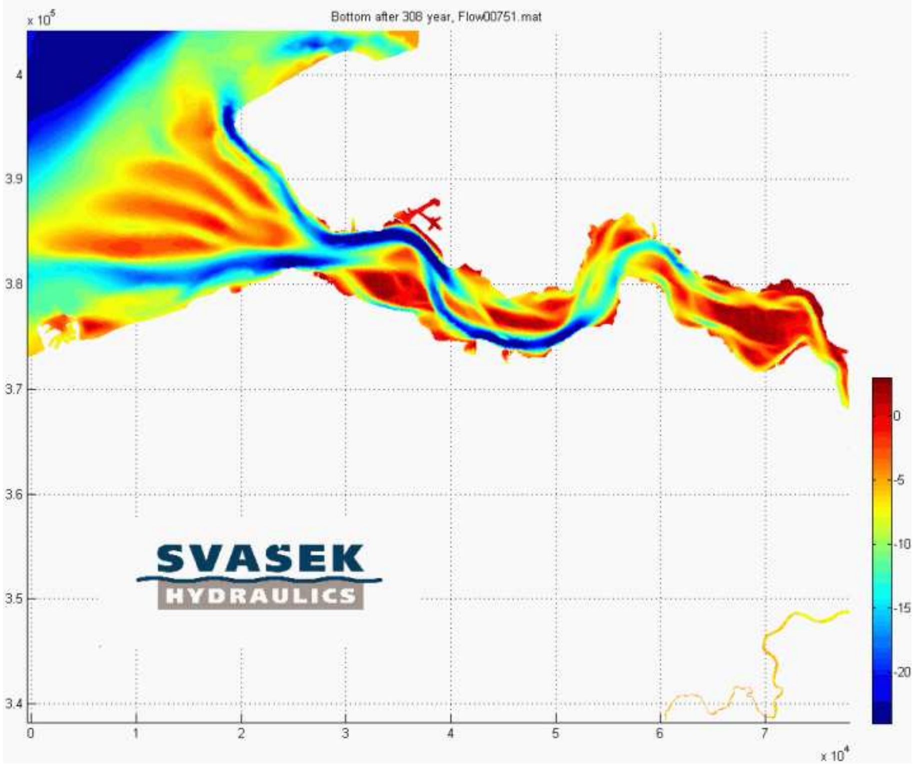

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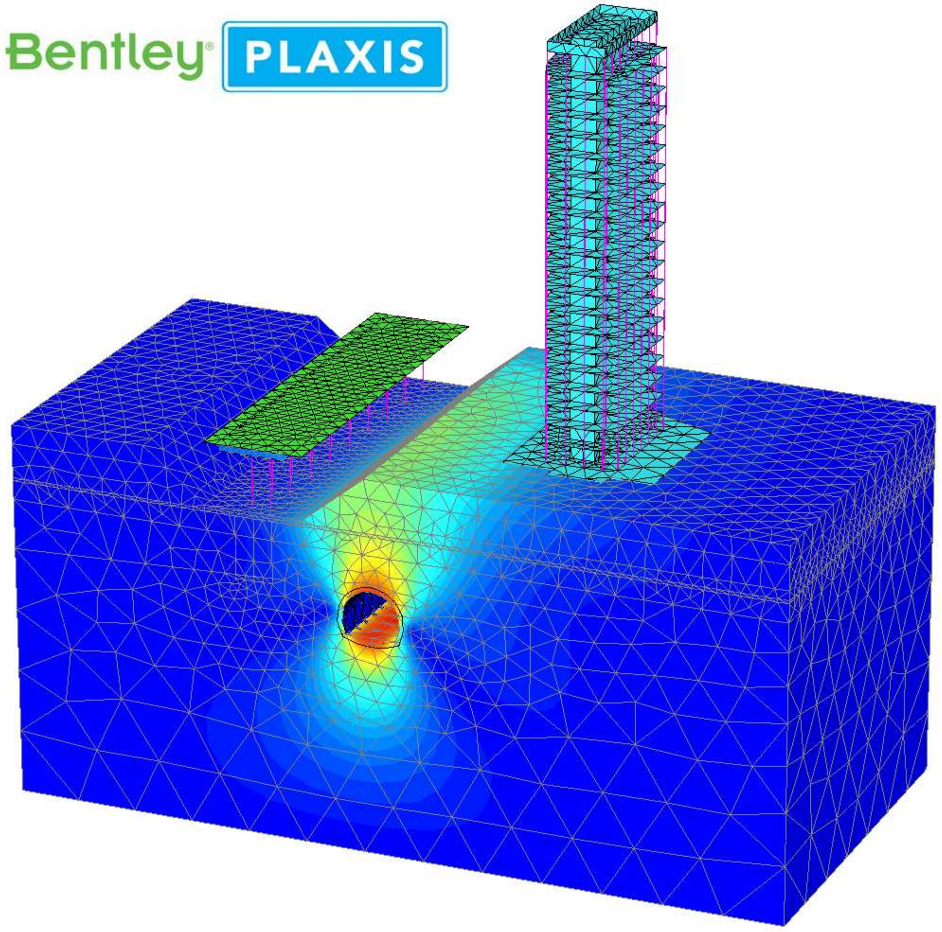

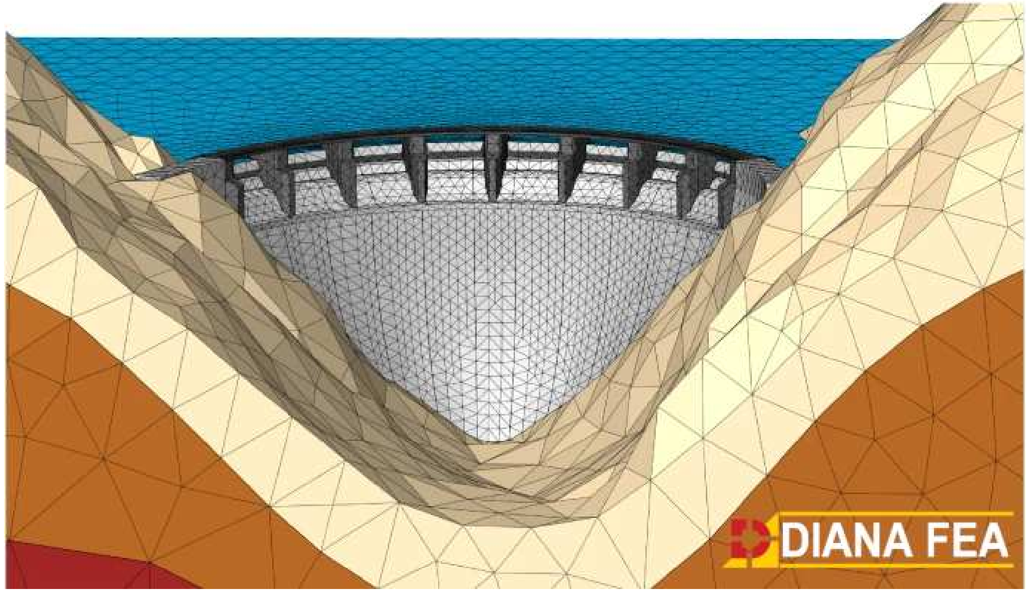

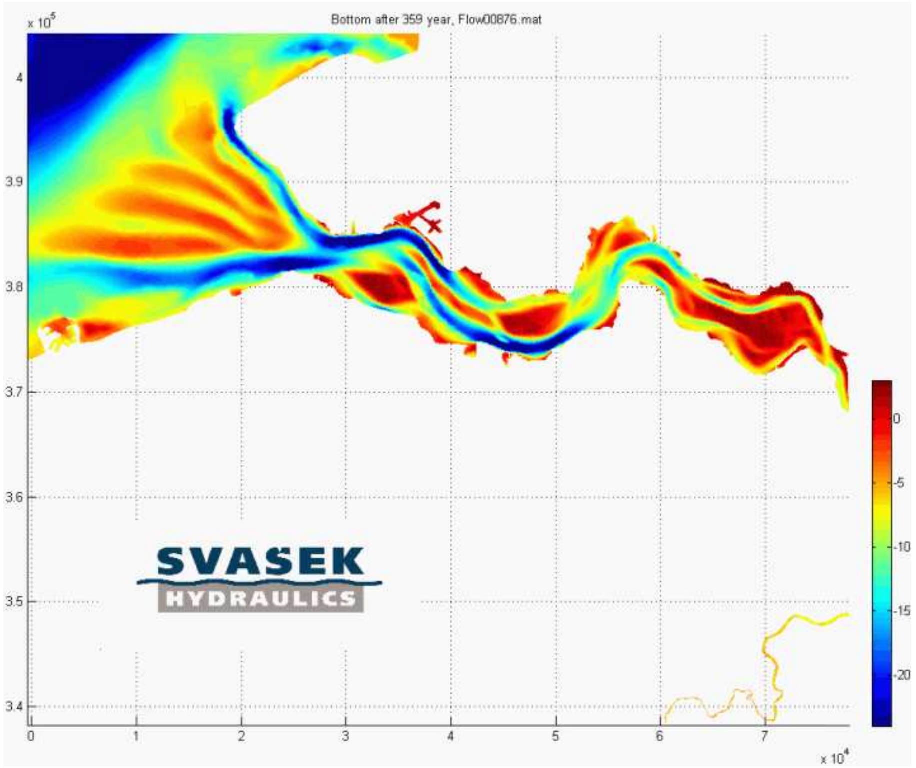

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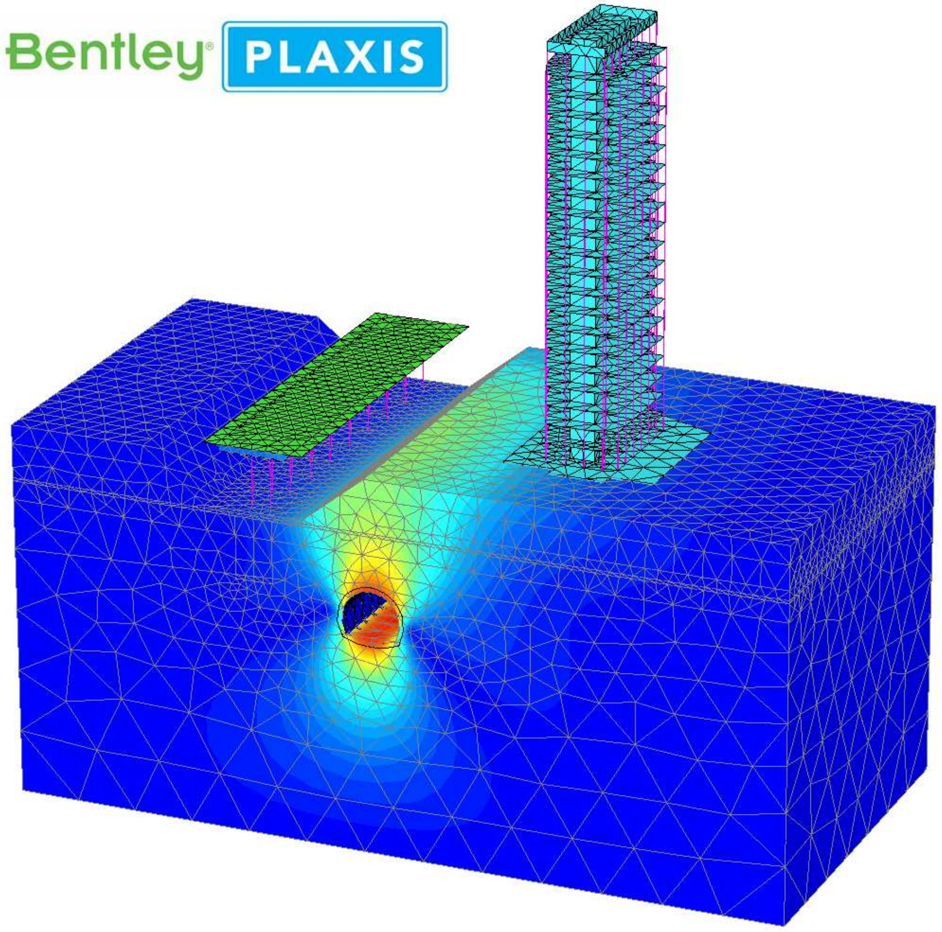

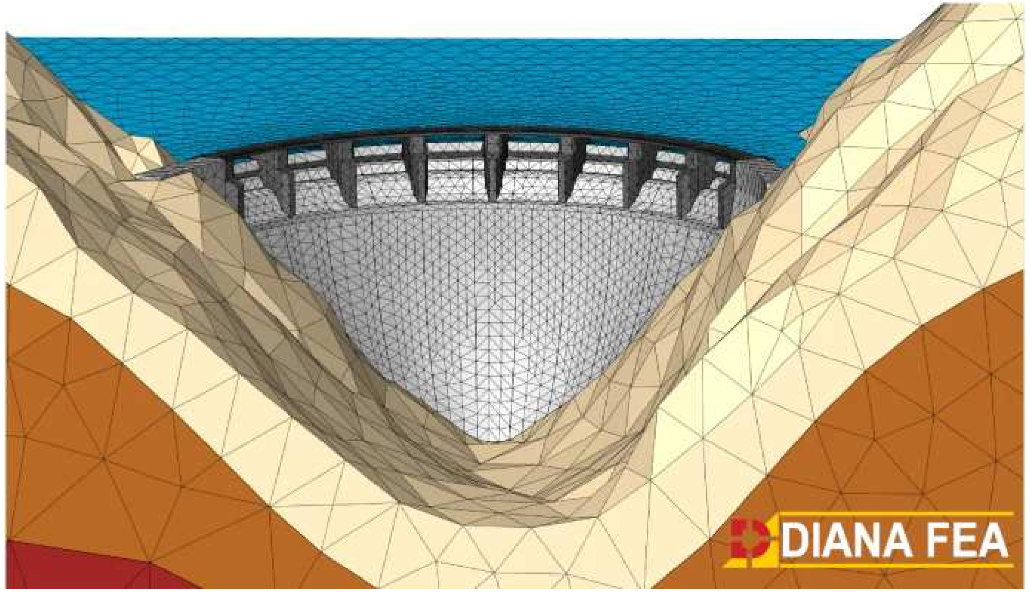

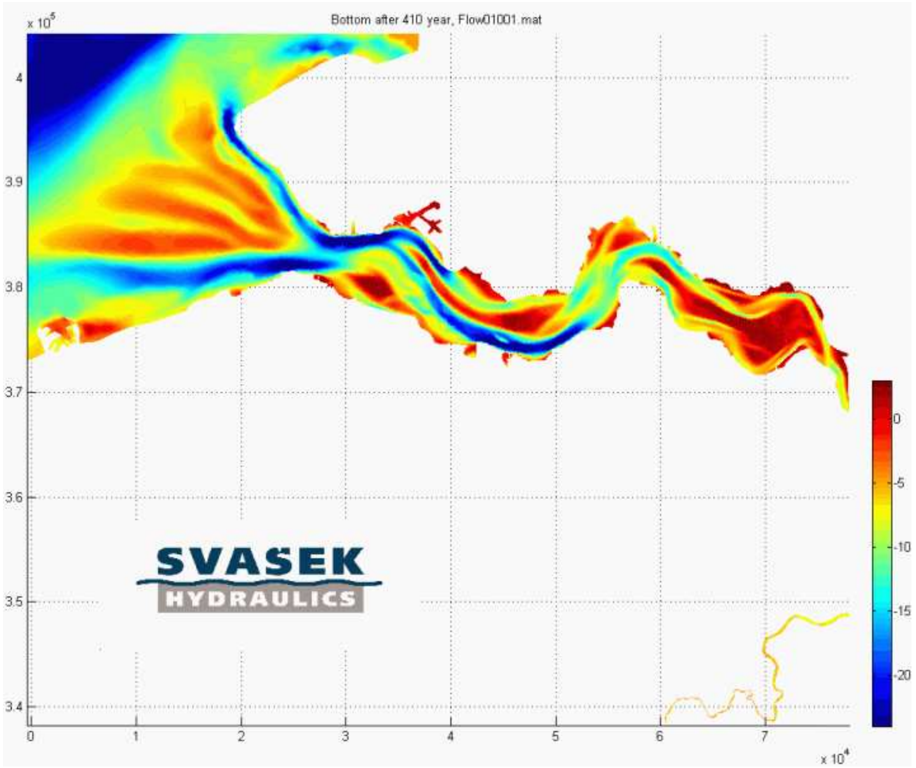

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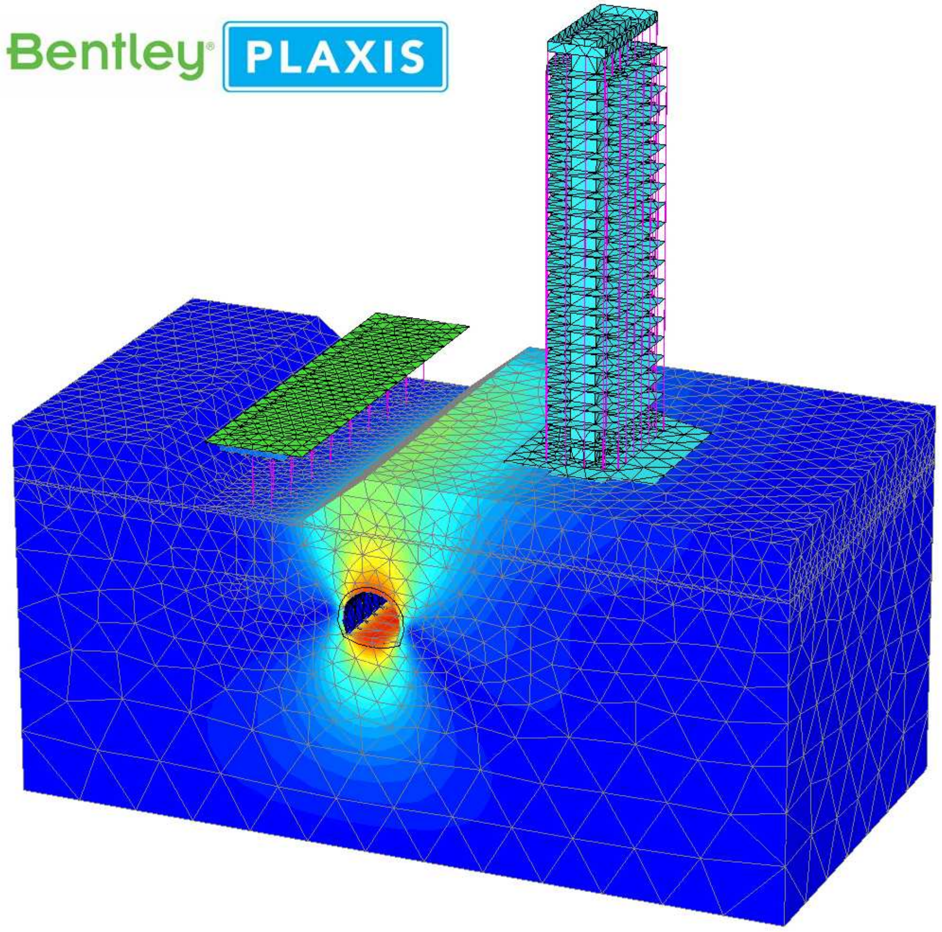

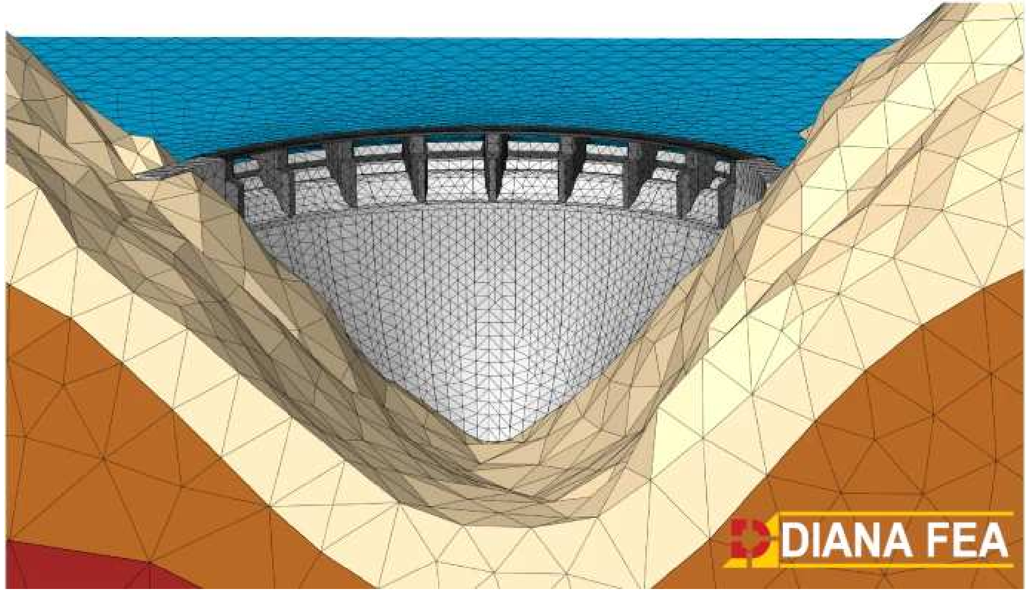

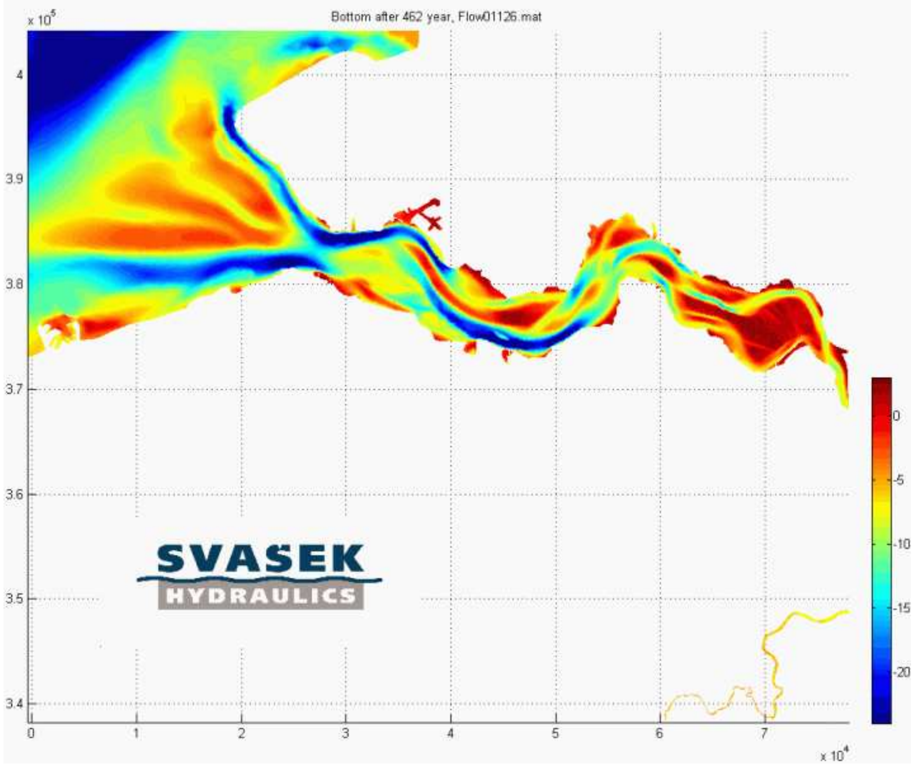

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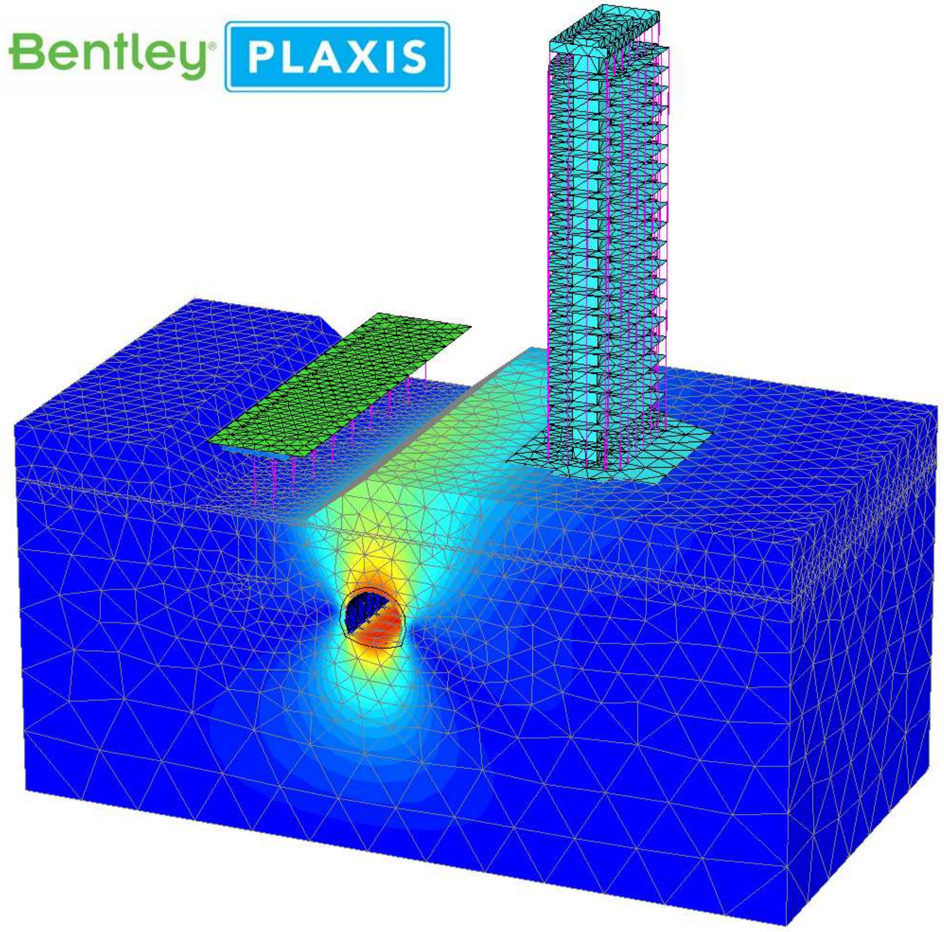

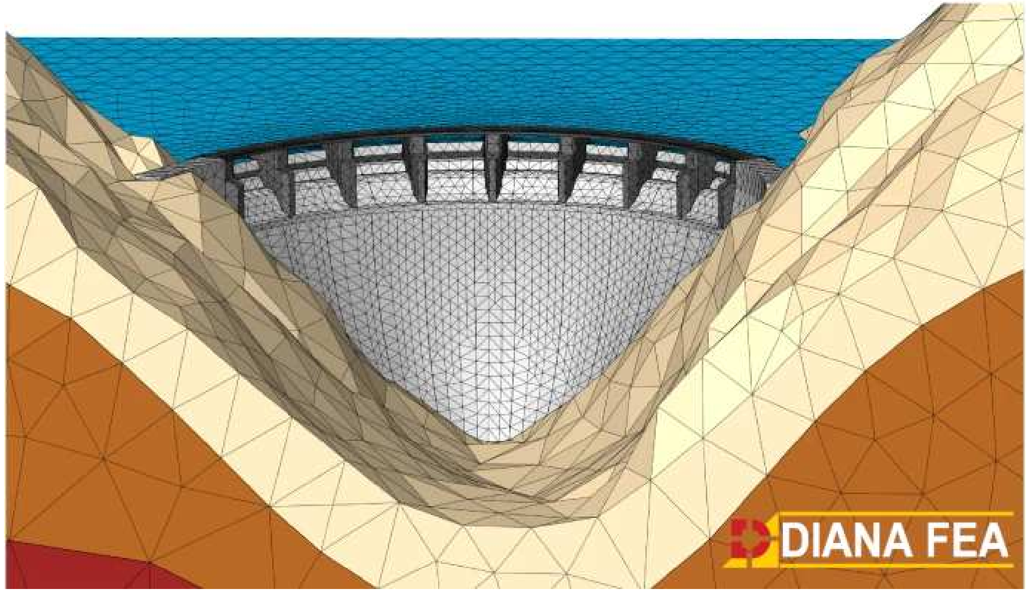

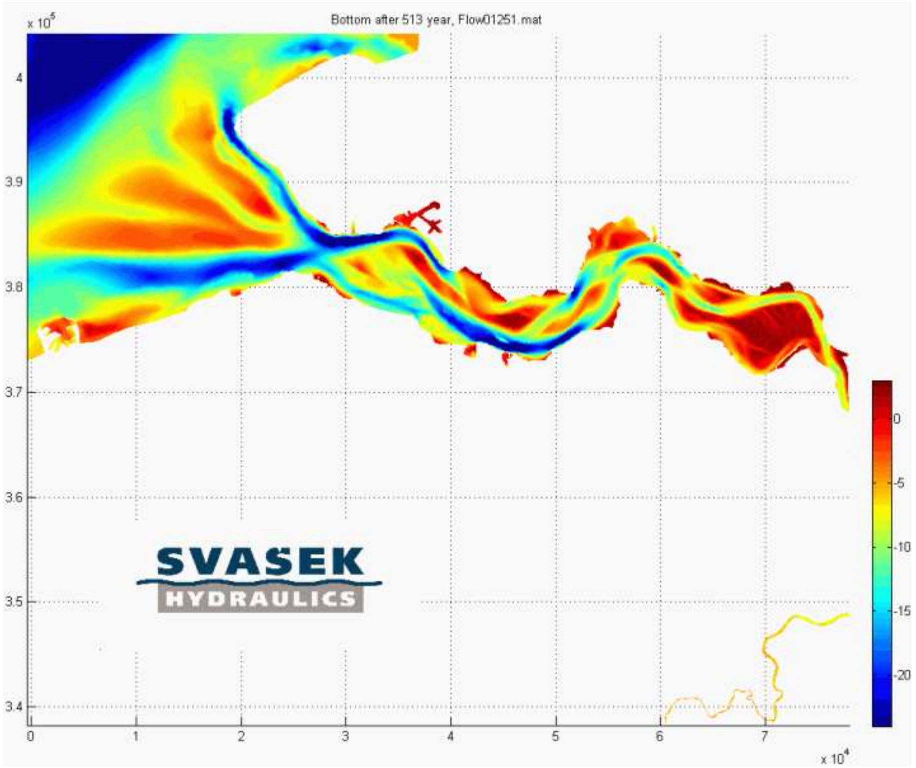

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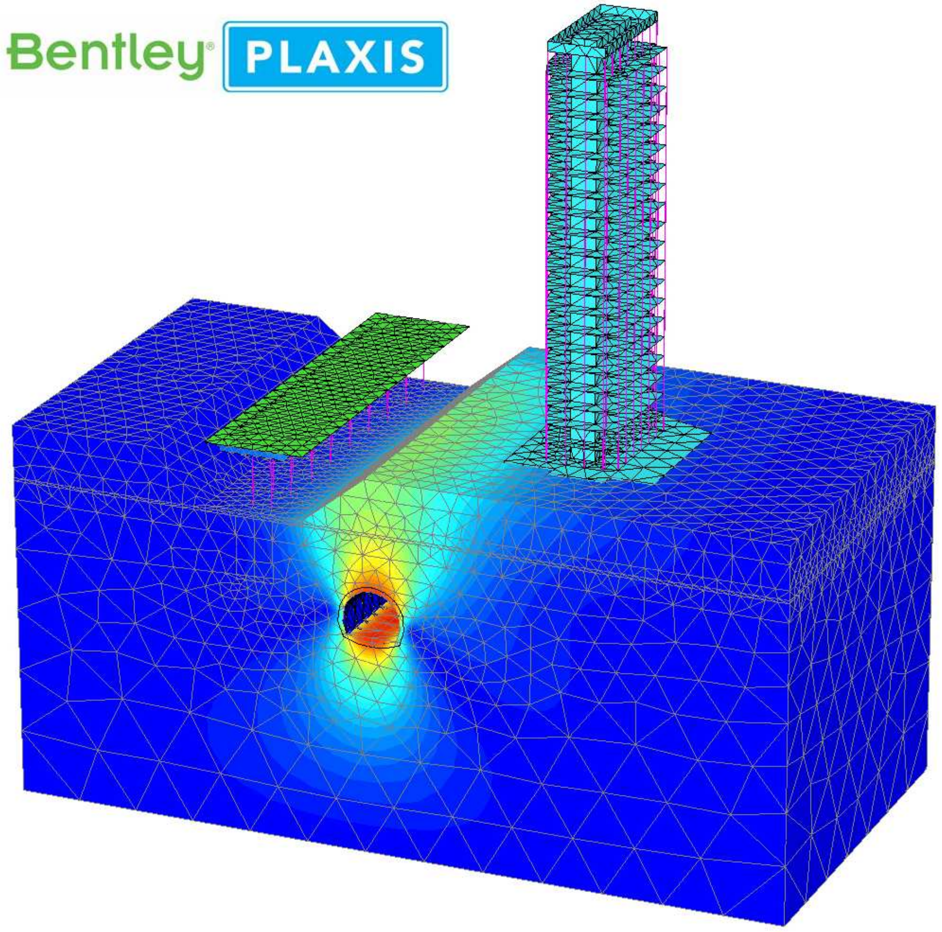

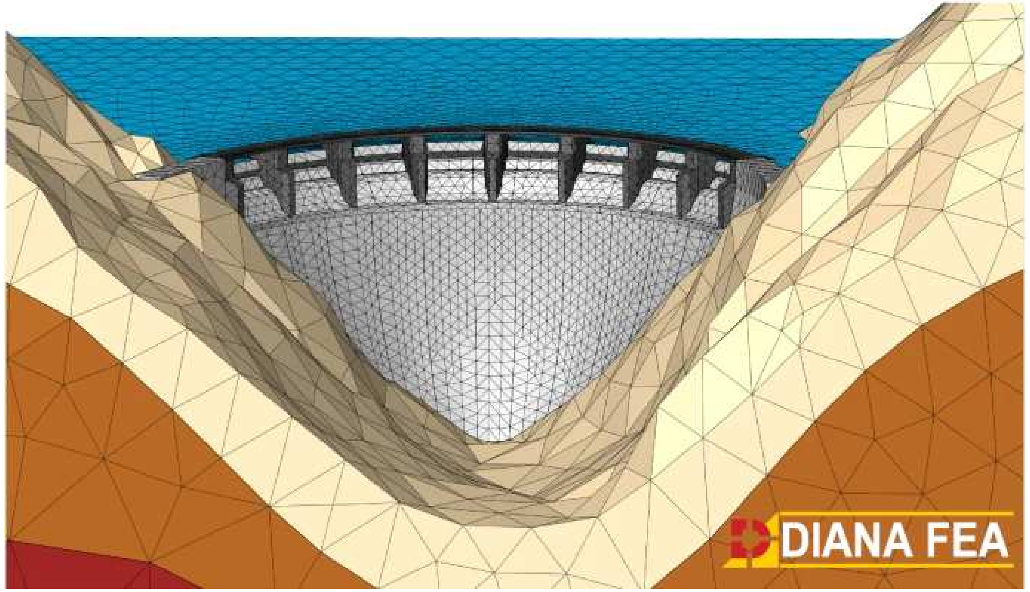

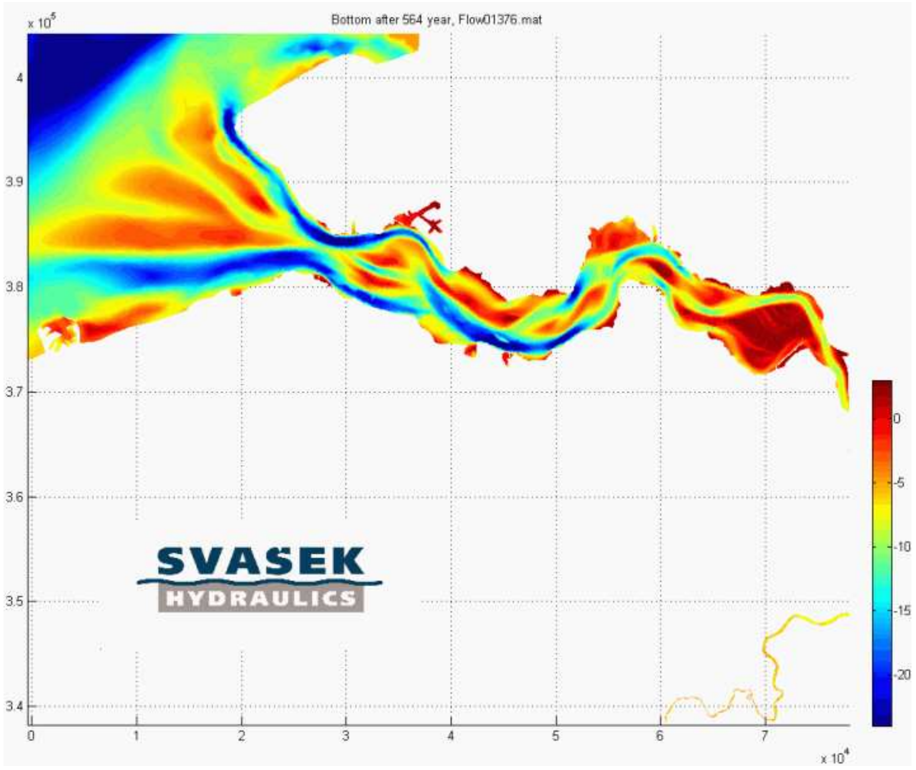

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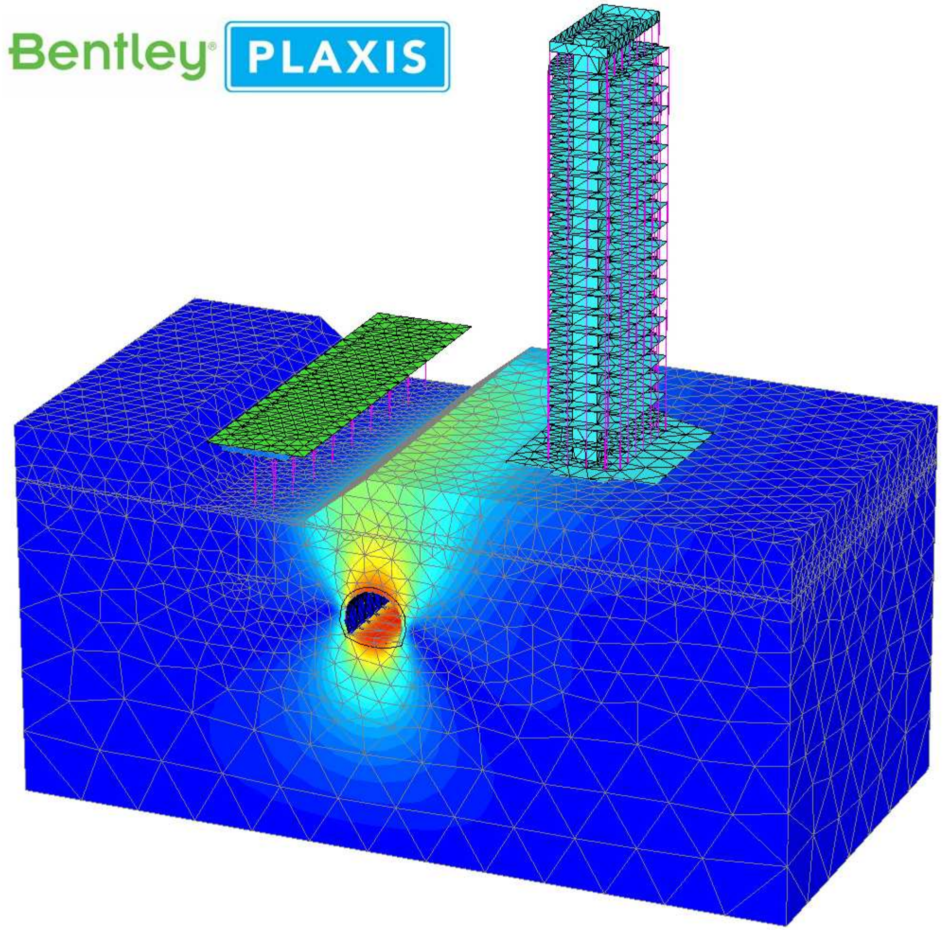

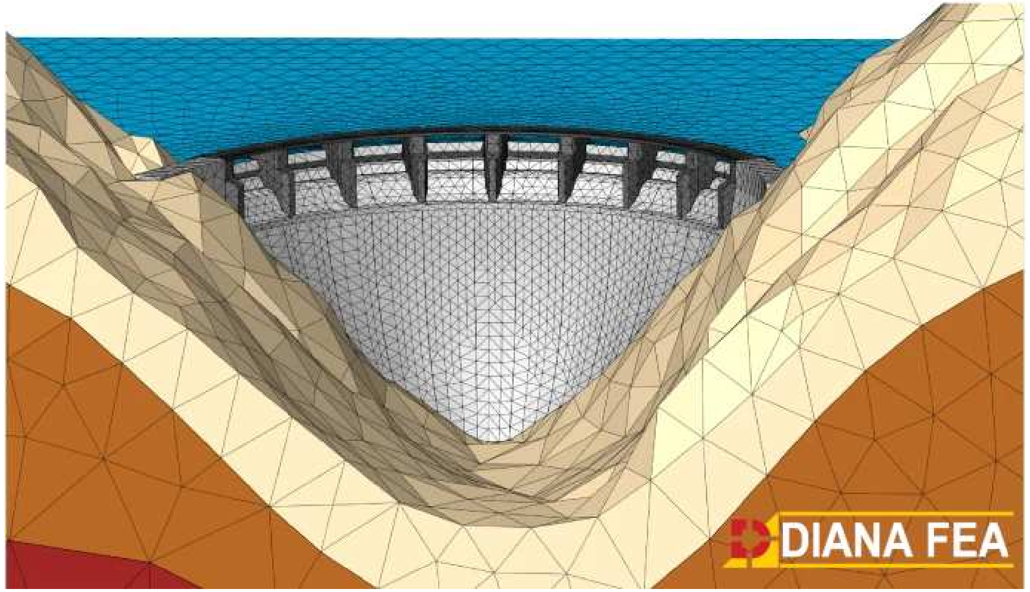

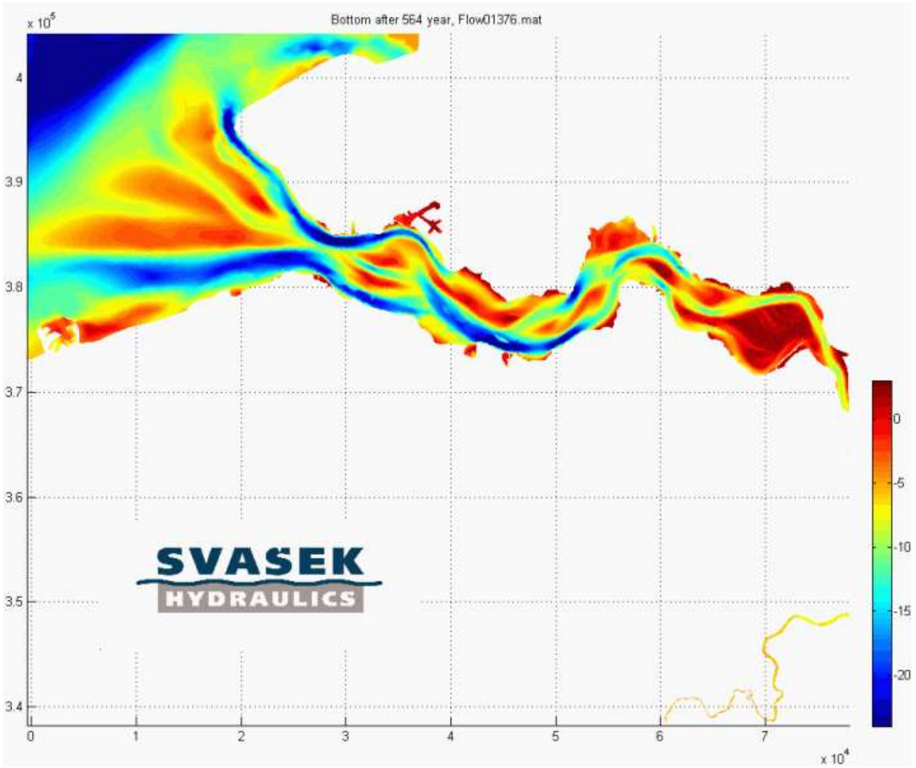

And several research codes

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The Finite __________ Methods

Finite difference method: discretize the derivatives

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$$

Finite volume method: discretize the conservation

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u \qquad \leadsto \qquad \frac{\partial}{\partial t}\int_\Omega u \,\mathrm{d}\Omega = \nu \int_\Gamma \nabla u \cdot \mathbf{n} \,\mathrm{d}\Gamma$$

Finite element method: discretize the solution

$$u(x) \approx \sum_i N_i(x) u_i$$

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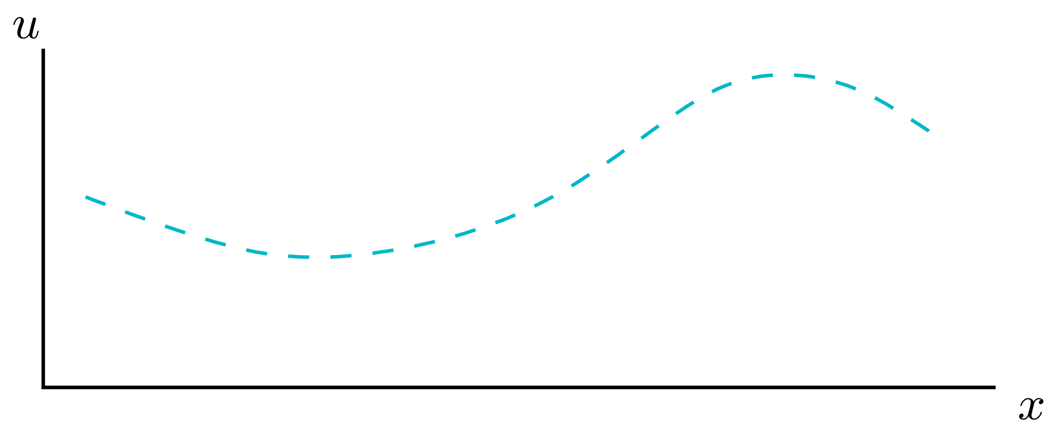
Discretizing the solution

The Poisson equation in 1D

$$-\nu \frac{\partial^2 u}{\partial x^2} = f$$



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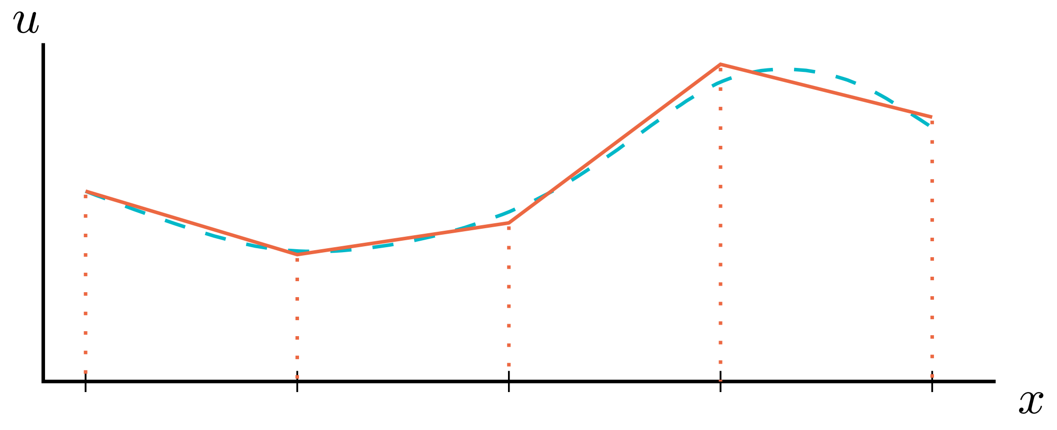

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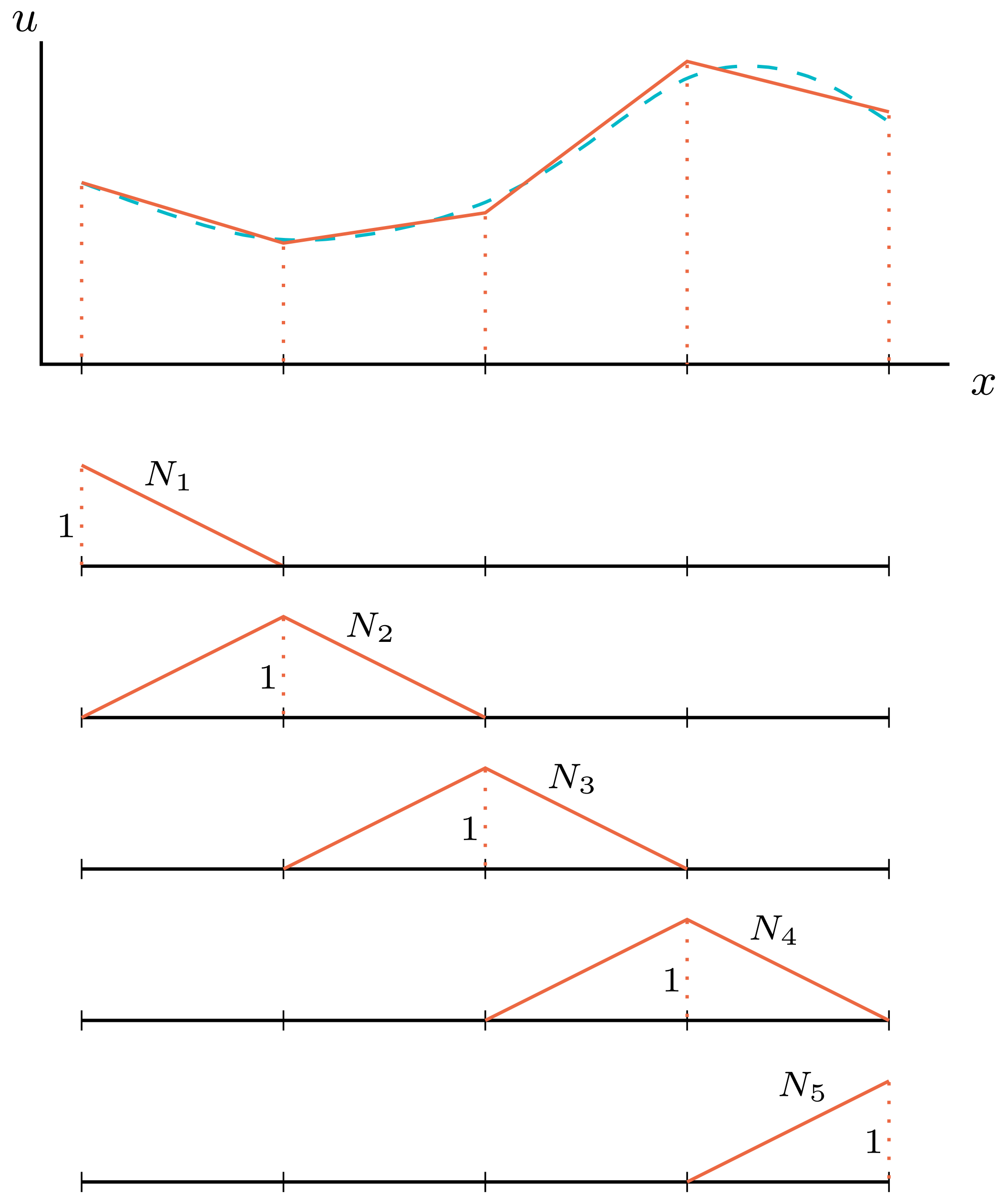

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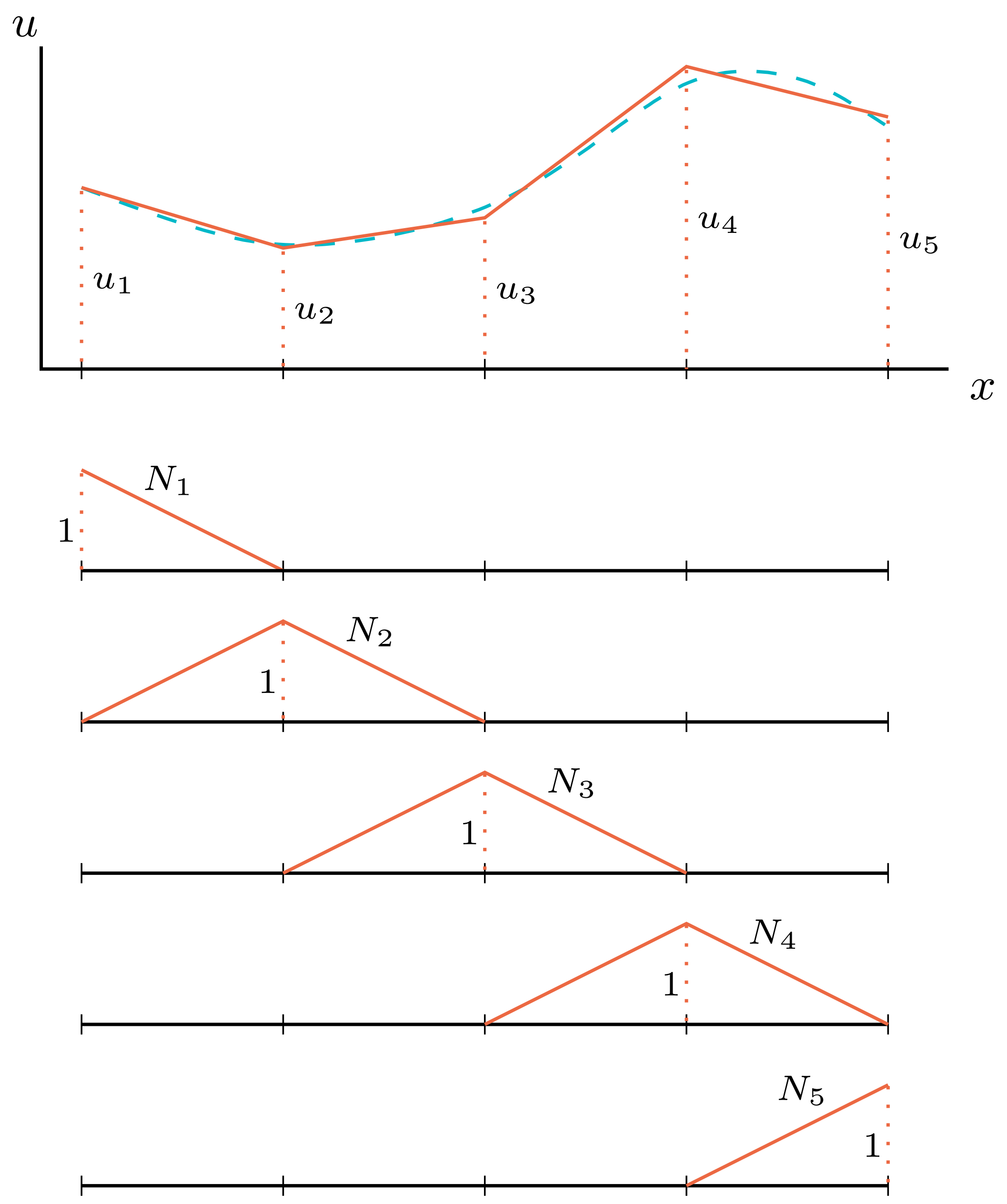

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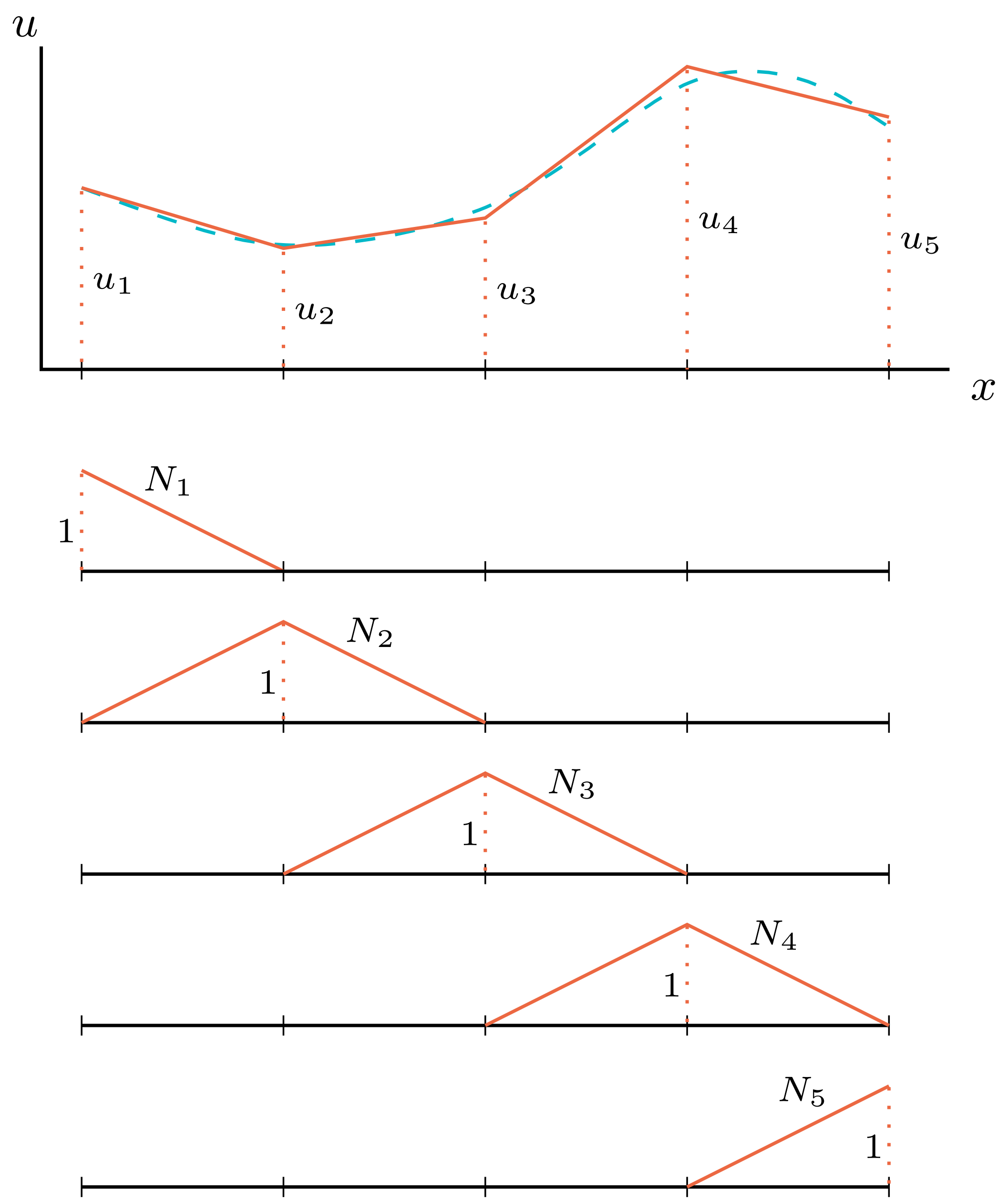
How to find the *best* values u_i?



From strong form to weak form equation

Weighted residual formulation:

$$-\nu\frac{\partial^2 u}{\partial x^2} = f \qquad \Leftrightarrow \qquad -\int_\Omega w\nu\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x = \int_\Omega wf\,\mathrm{d}x \quad \forall \quad w$$

Integration by parts:

$$\int_\Omega w\nu\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x = -\int_\Omega \frac{\partial w}{\partial x}\nu\frac{\partial u}{\partial x}\,\mathrm{d}x + \left[w\nu\frac{\partial u}{\partial x}\right]_0^L$$

Substitution of boundary conditions:

$$\int_\Omega \frac{\partial w}{\partial x}\nu\frac{\partial u}{\partial x}\,\mathrm{d}x = \int_\Omega wf\,\mathrm{d}x + w(L)h(L) - w(0)h(0) \quad \forall \quad w$$

From weak form to discretized form

Weak form equation

$$\int_\Omega \frac{\partial w}{\partial x}\nu\frac{\partial u}{\partial x}\,\mathrm{d}x = \int_\Omega wf\,\mathrm{d}x + [wh]_0^L \quad \forall \quad w$$

Introduce discretization:

$$u \leftarrow u^h = \mathbf{Nu}, \qquad w \leftarrow w^h = \mathbf{Nw}$$ (Bubnov-Galerkin)

$$\frac{\partial u}{\partial x} \leftarrow \frac{\partial u^h}{\partial x} = \mathbf{Bu}, \qquad \frac{\partial w}{\partial x} \leftarrow \frac{\partial w^h}{\partial x} = \mathbf{Bw}$$

Substitution gives:

$$\int_\Omega \mathbf{Bw}\nu\mathbf{Bu}\,\mathrm{d}x = \int_\Omega \mathbf{Nw}f\,\mathrm{d}x + [wh]_0^L \quad \forall \quad \mathbf{w} \qquad \Rightarrow \qquad \int_\Omega \mathbf{B}^T\nu\mathbf{B}\,\mathrm{d}x\,\mathbf{u} = \int_\Omega \mathbf{N}^T f\,\mathrm{d}x + \left[\mathbf{N}^T h\right]_0^L$$

The resulting system of equations

$$\mathbf{K}\mathbf{u} = \mathbf{f} \quad \text{with} \quad \mathbf{K} = \int_\Omega \mathbf{B}^T \nu \mathbf{B} \, \mathrm{d}x \quad \text{and} \quad \mathbf{f} = \int_\Omega \mathbf{N}^T f \, \mathrm{d}x + \left[\mathbf{N}^T h\right]_0^L$$

expanded as:

$$\mathbf{K} = \int_\Omega \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \nu \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix} \mathrm{d}x$$

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with: $$N_i = \begin{cases} 0, & x \leq x_{i-1} \\ \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \leq x < x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i \leq x < x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

and: $$B_i = \frac{\partial N_i}{\partial x} = \begin{cases} 0, & x \leq x_{i-1} \\ \frac{1}{x_i - x_{i-1}}, & x_{i-1} \leq x < x_i \\ \frac{-1}{x_{i+1} - x_i}, & x_i \leq x < x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

After integration:

$$\frac{\nu}{\Delta x} \begin{bmatrix} 1 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & & 0 & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 & 0 \\ & & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 & \ddots & 2 & -1 & 0 \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} \frac{1}{2} q\Delta x - h(0) \\ q\Delta x \\ q\Delta x \\ \vdots \\ q\Delta x \\ q\Delta x \\ \frac{1}{2} q\Delta x + h(L) \end{bmatrix}$$

(with uniform mesh $x_{i+1} - x_i = \Delta x$
and constant source $f(x) = q$)

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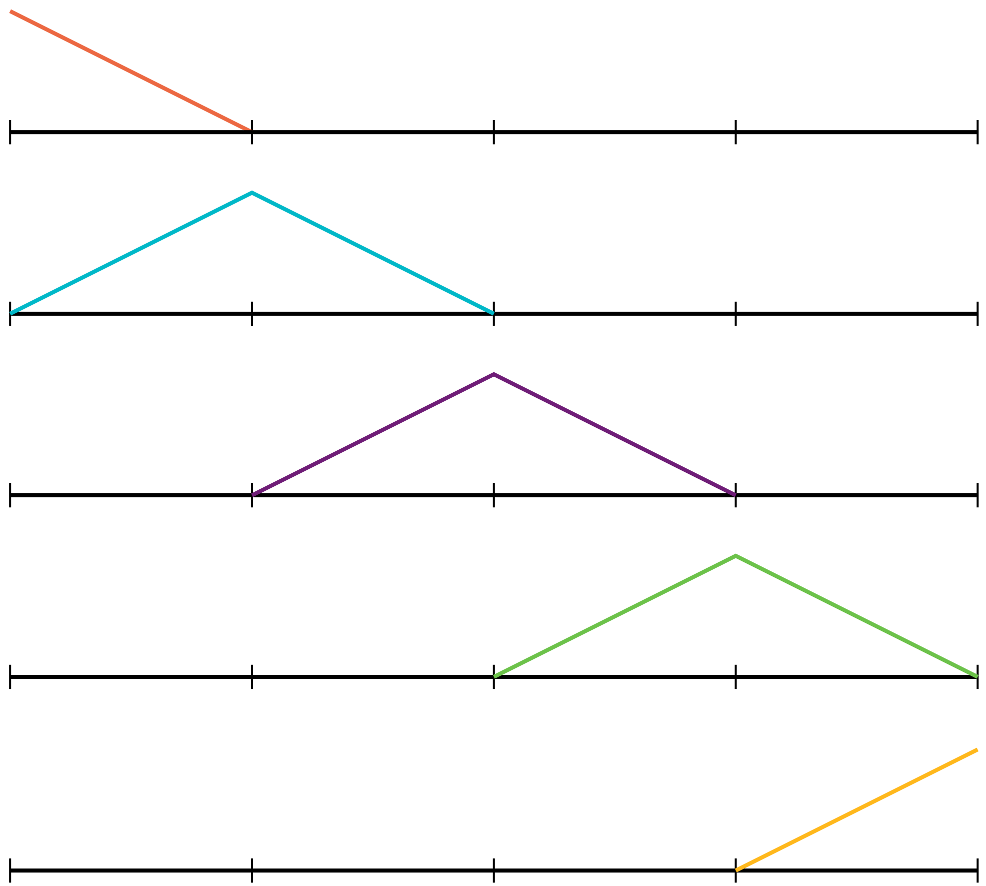
Now, what about these 'Elements'?

The discretized Poisson equation: $\mathbf{K}\mathbf{a} = \mathbf{f}$ with $\mathbf{K} = \int_\Omega \mathbf{B}^T \nu \mathbf{B}\, \mathrm{d}x$ and $\mathbf{f} = \int_\Omega \mathbf{N}^T f\, \mathrm{d}x + \left[\mathbf{N}^T h\right]_0^L$



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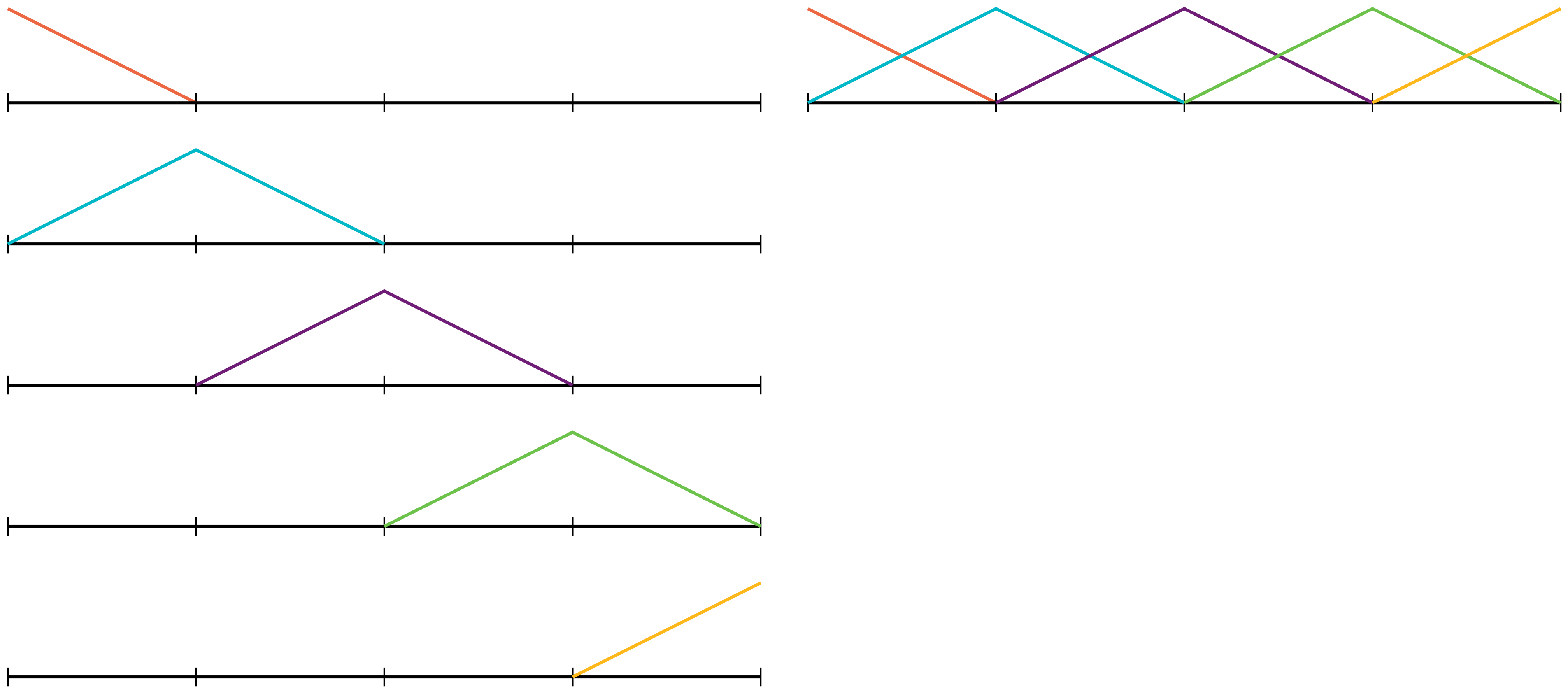

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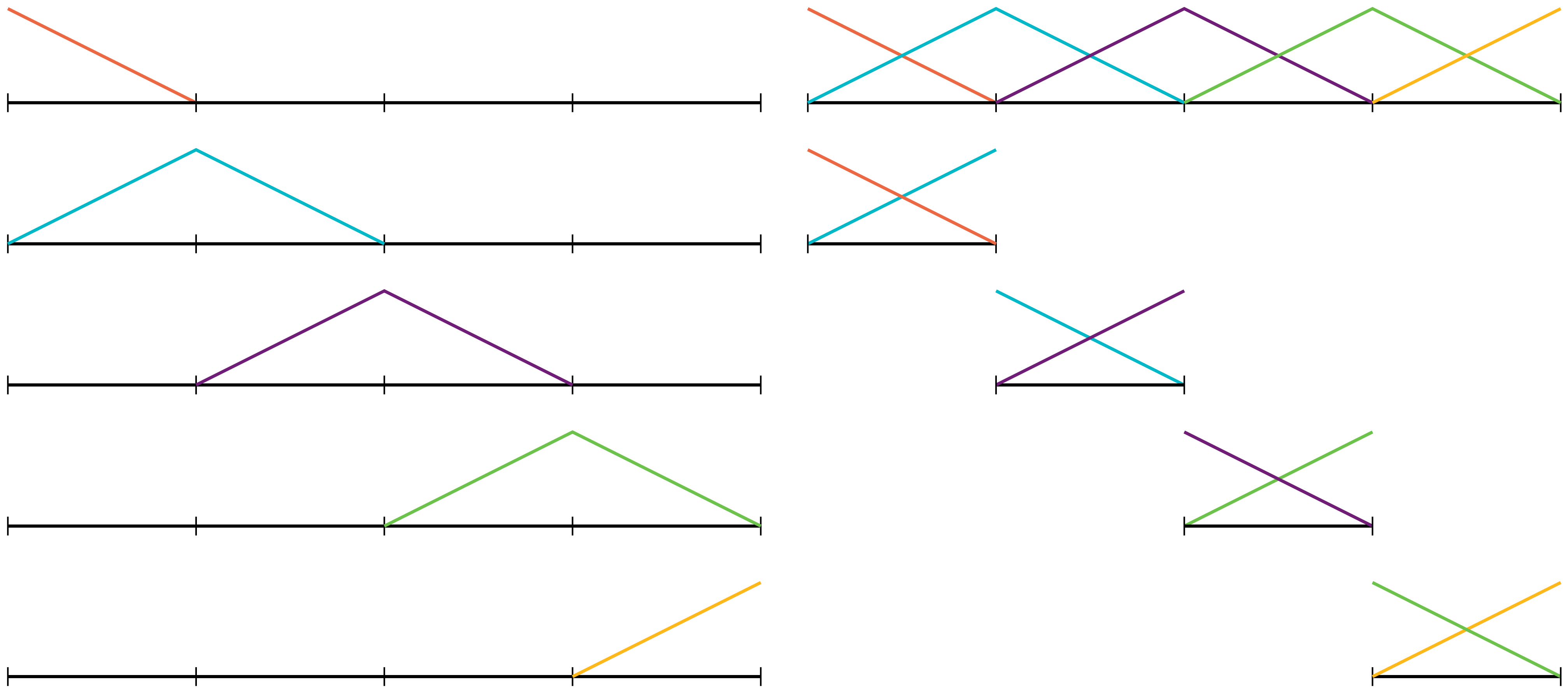

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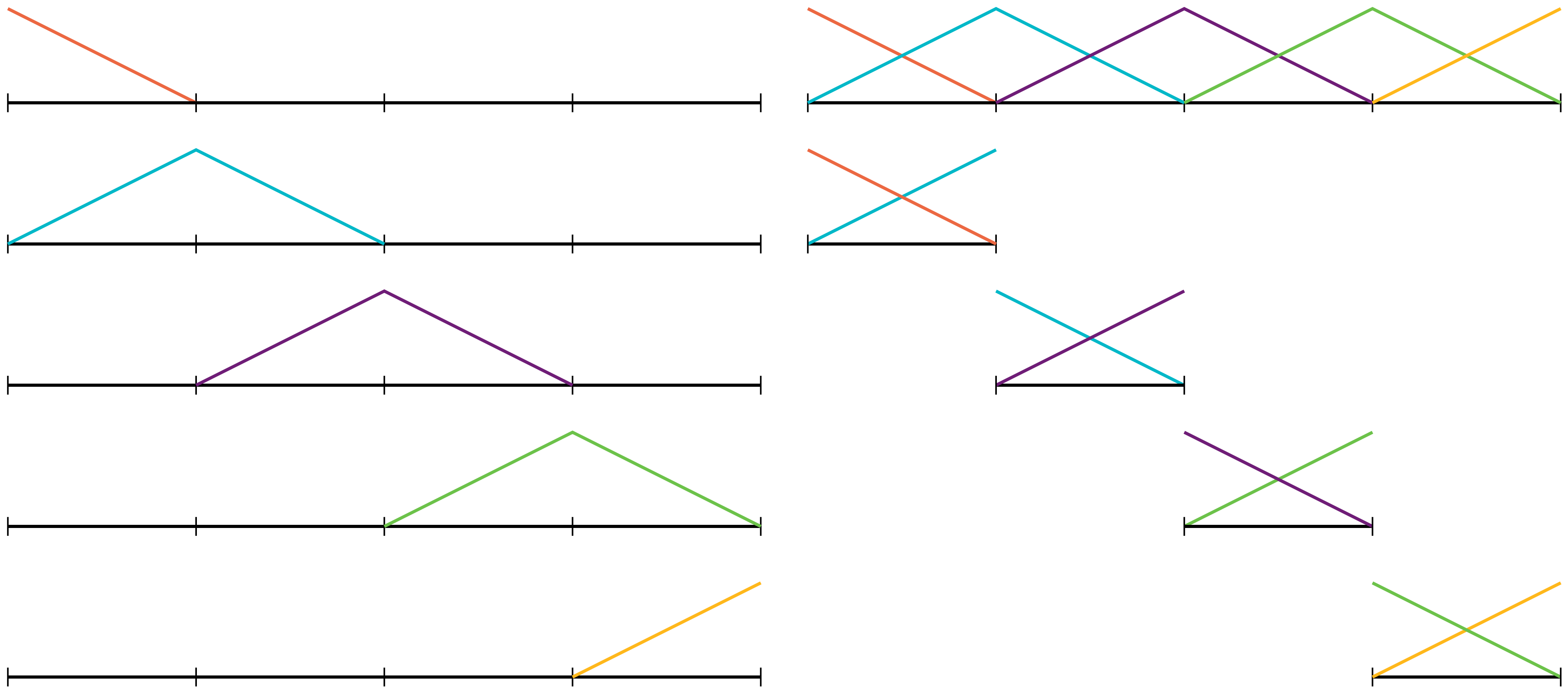

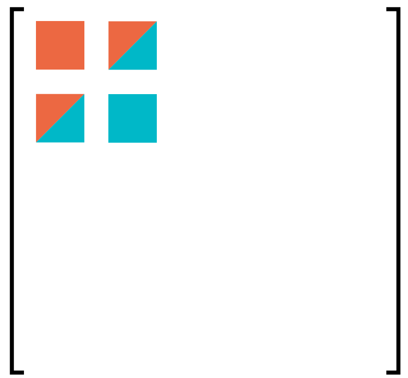

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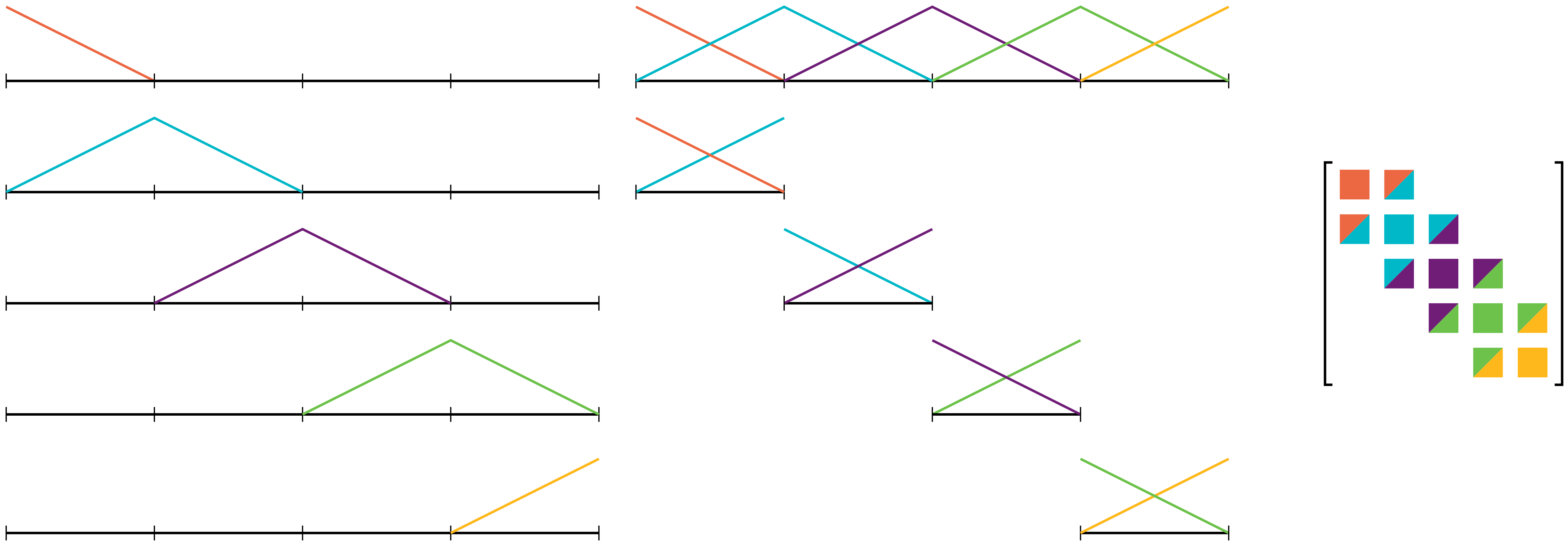

Higher order elements can also be formulated, 3-nodes per element for quadratic

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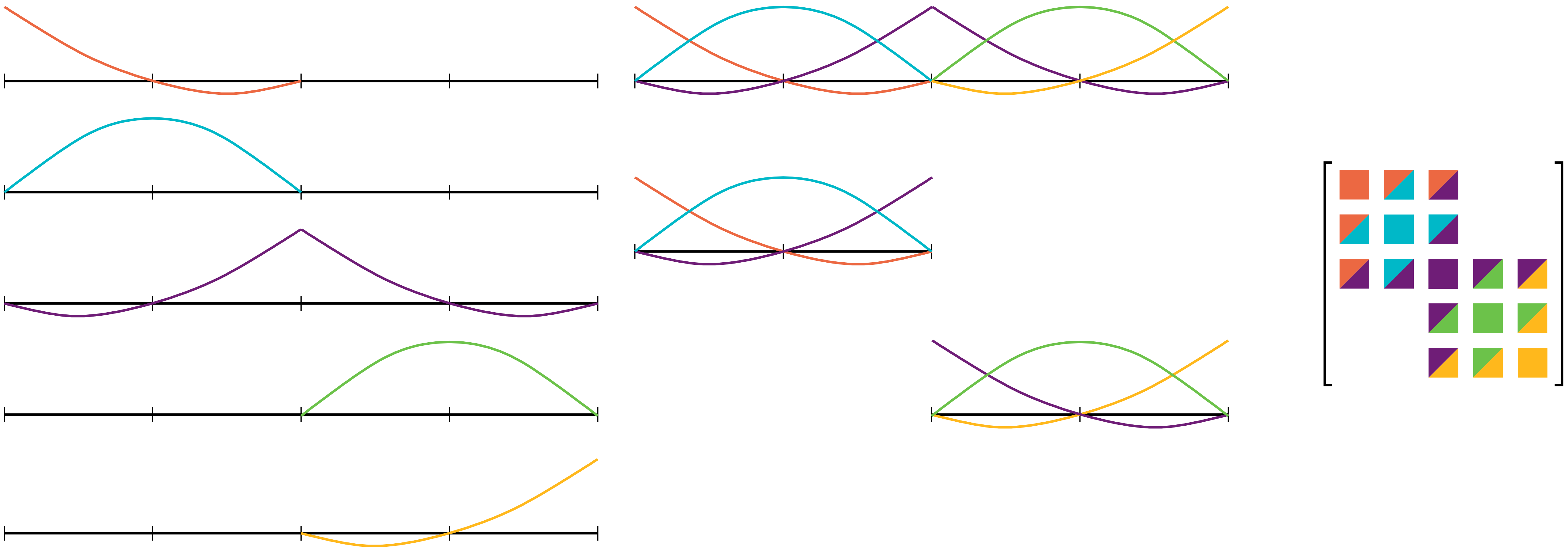


Shape function properties

Shape functions requirements for good performance:

- Partition of unity: $\sum_i N_i(x) = 1$

  $\Rightarrow$ represent constant solutions exactly

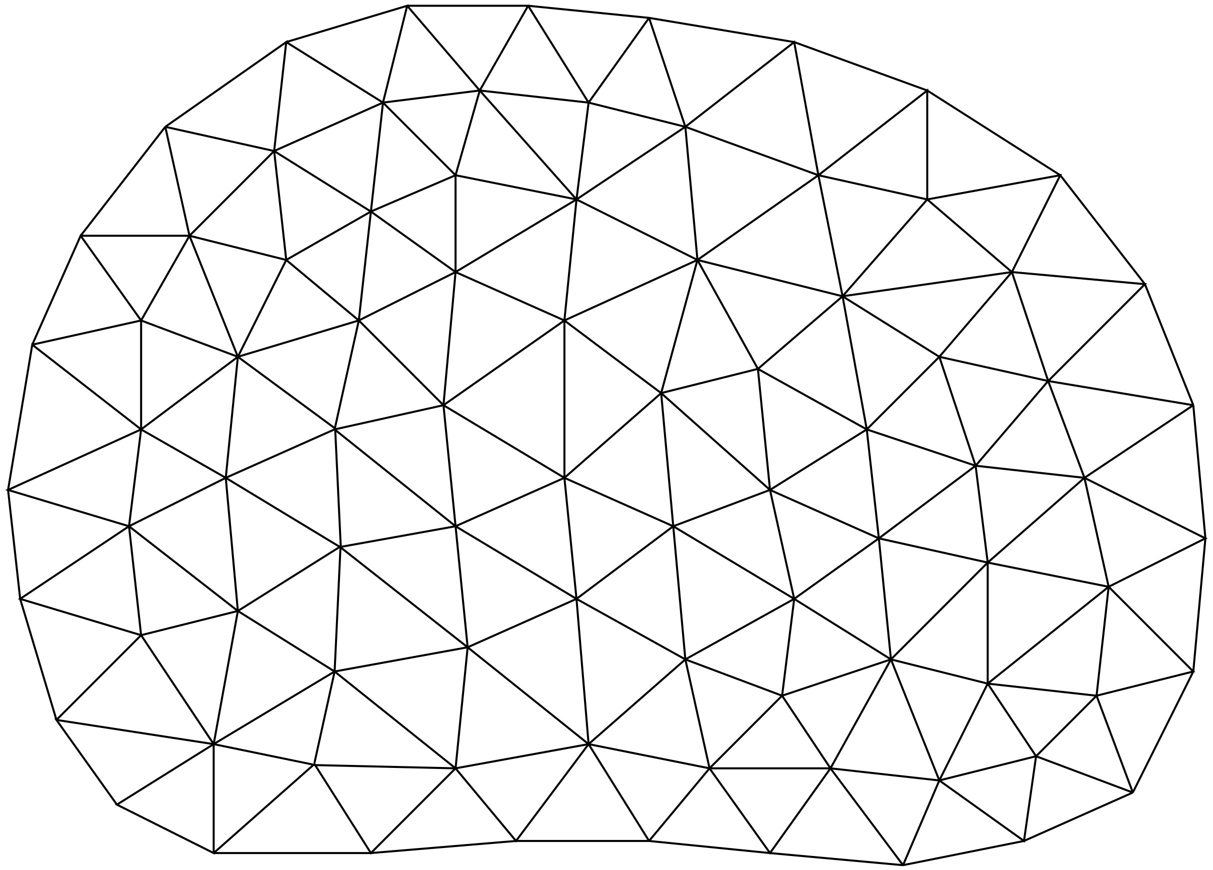
- Kronecker delta property: $N_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

  $\Rightarrow$ interpret degrees of freedom as nodal values

  $\Rightarrow$ apply boundary conditions directly

Discretizing a 2D solution with triangulation of the domain

$$u(x,y) \approx \sum_i N_i(x,y) u_i$$



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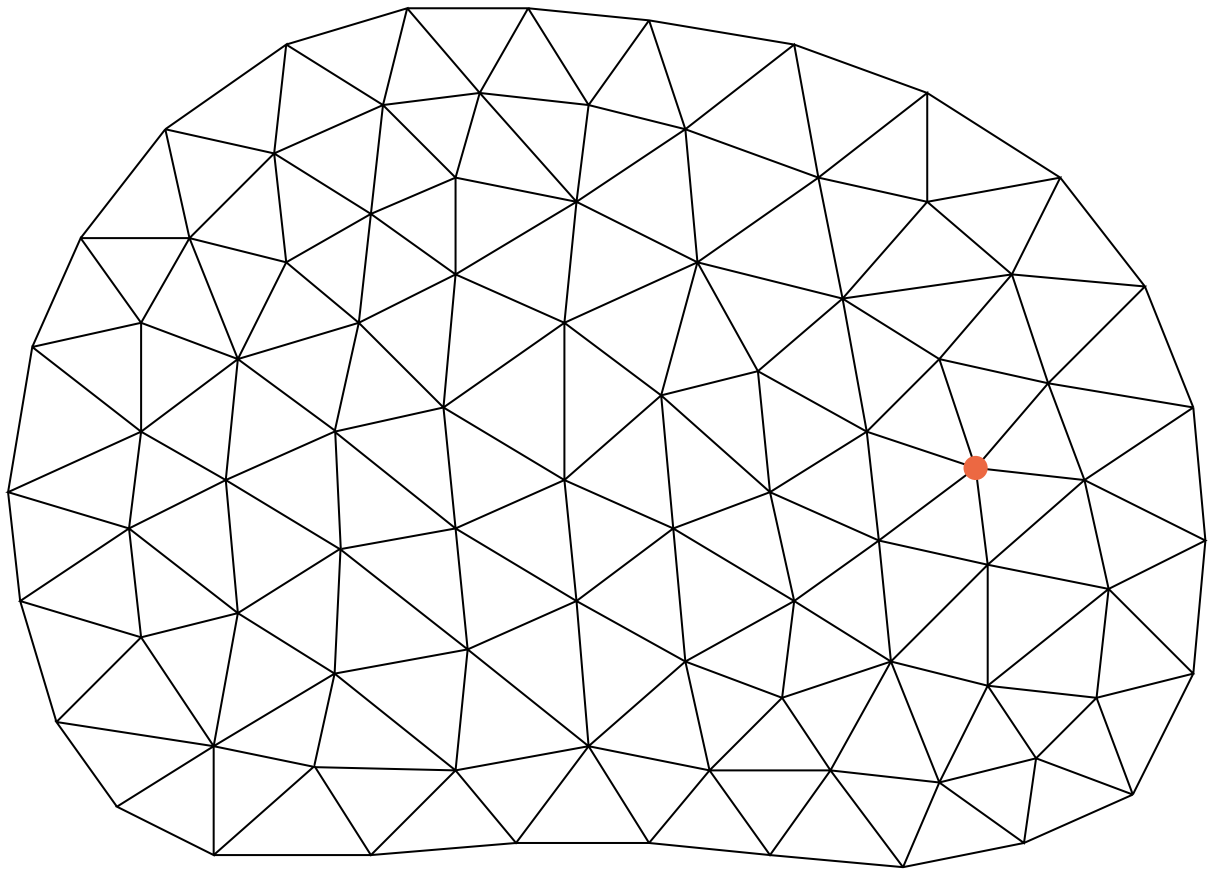

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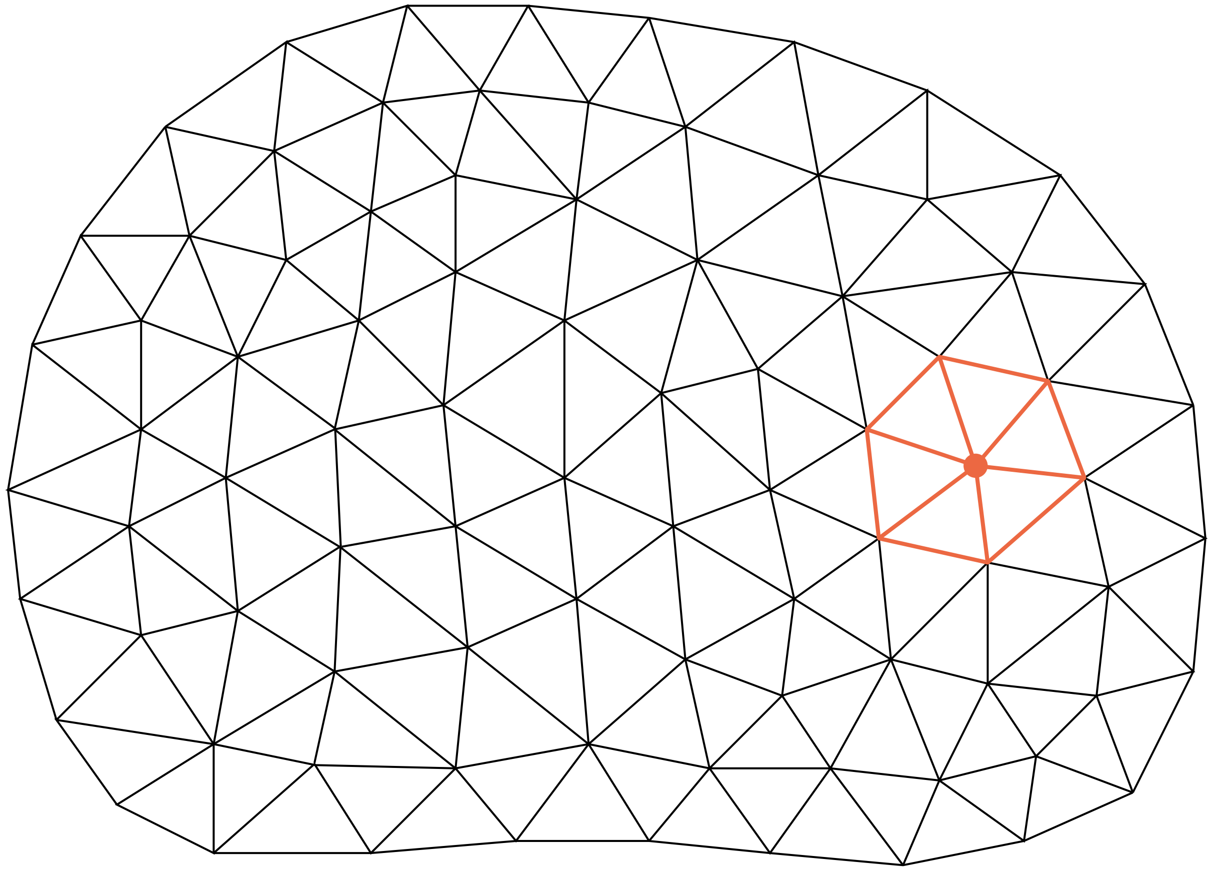

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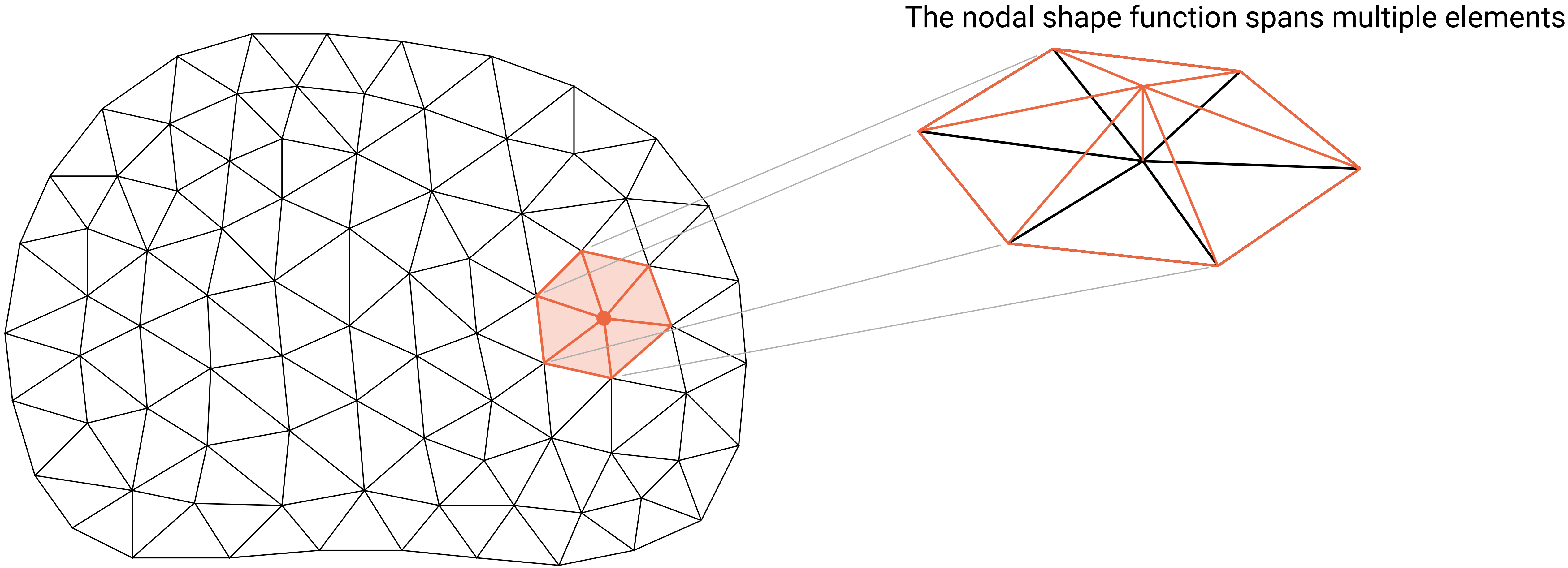

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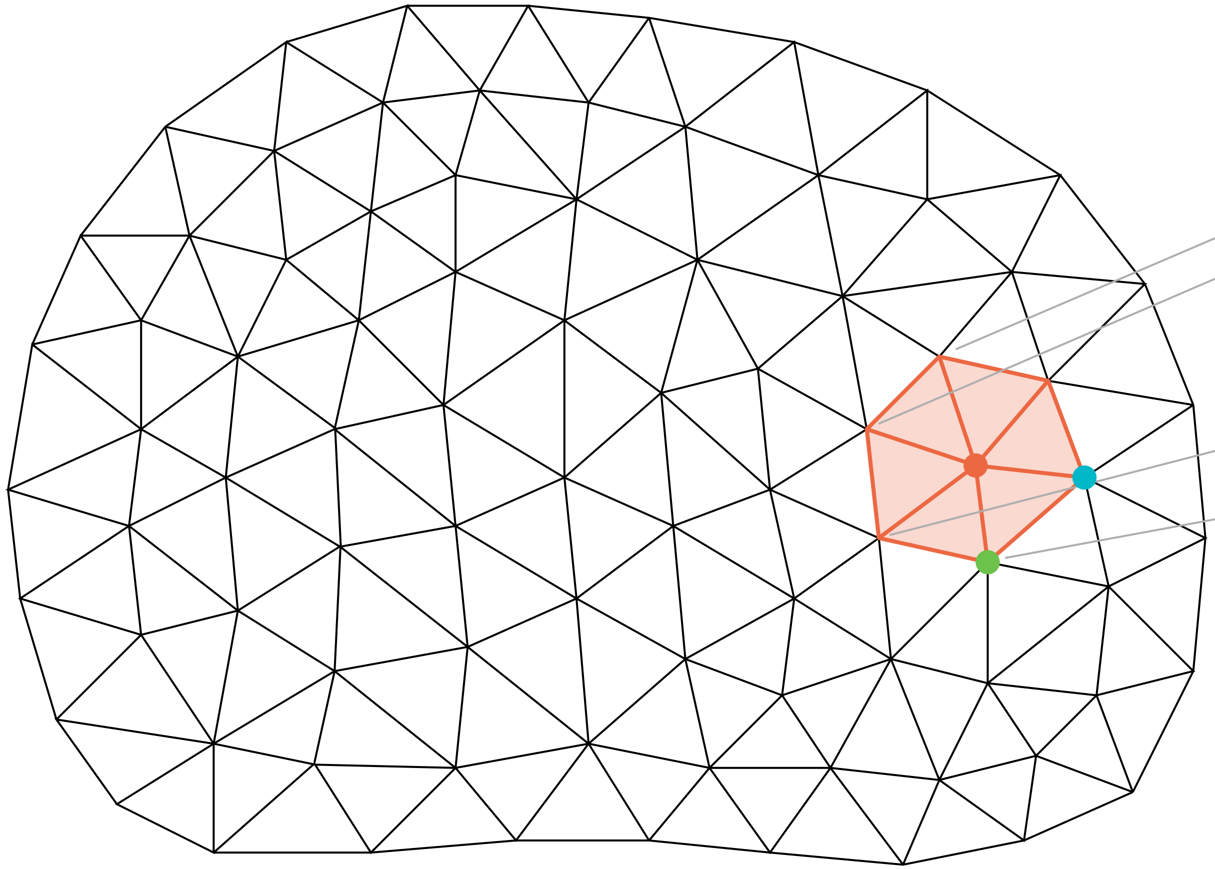
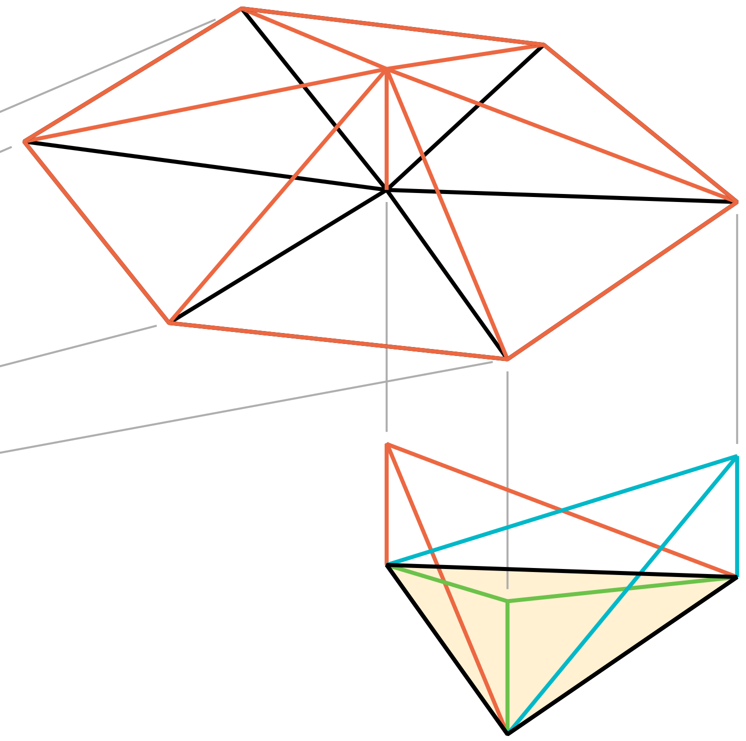


The nodal shape function spans multiple elements



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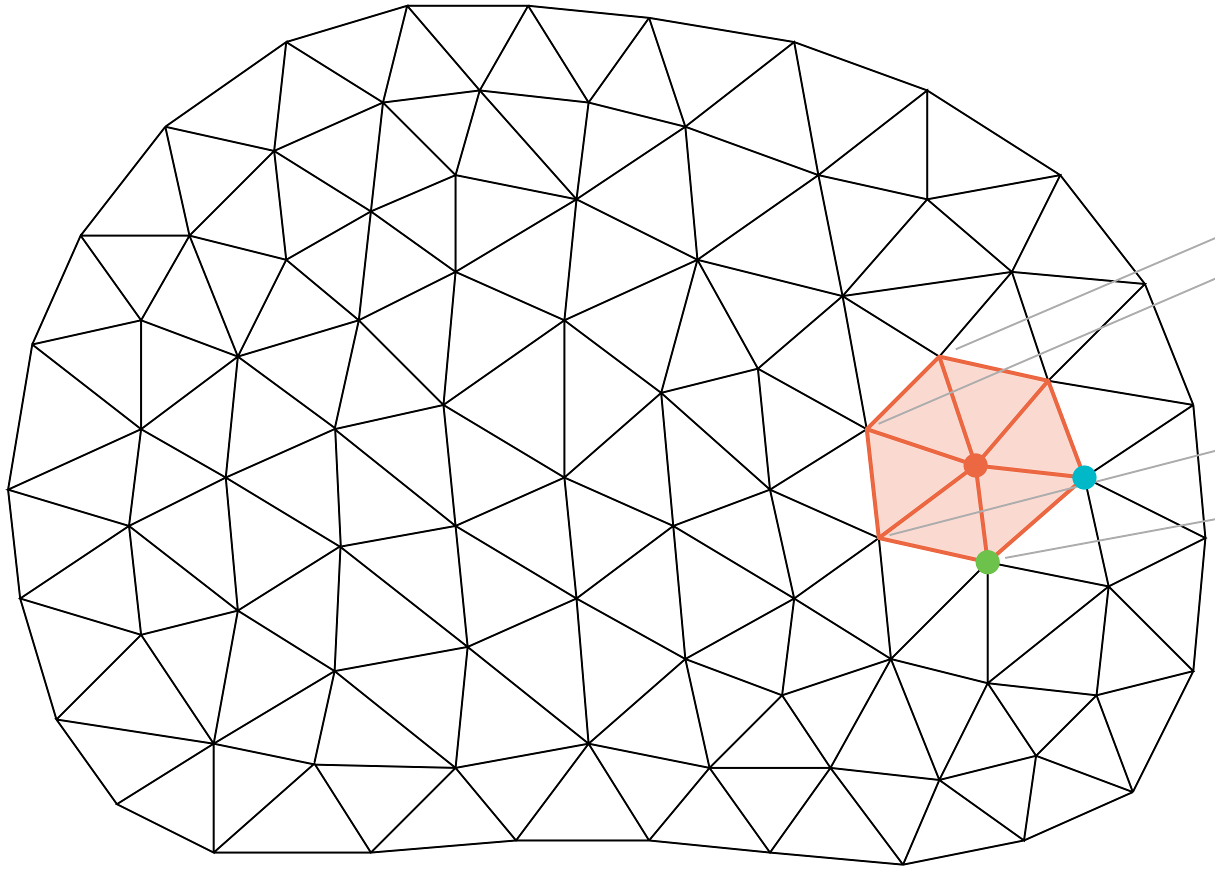

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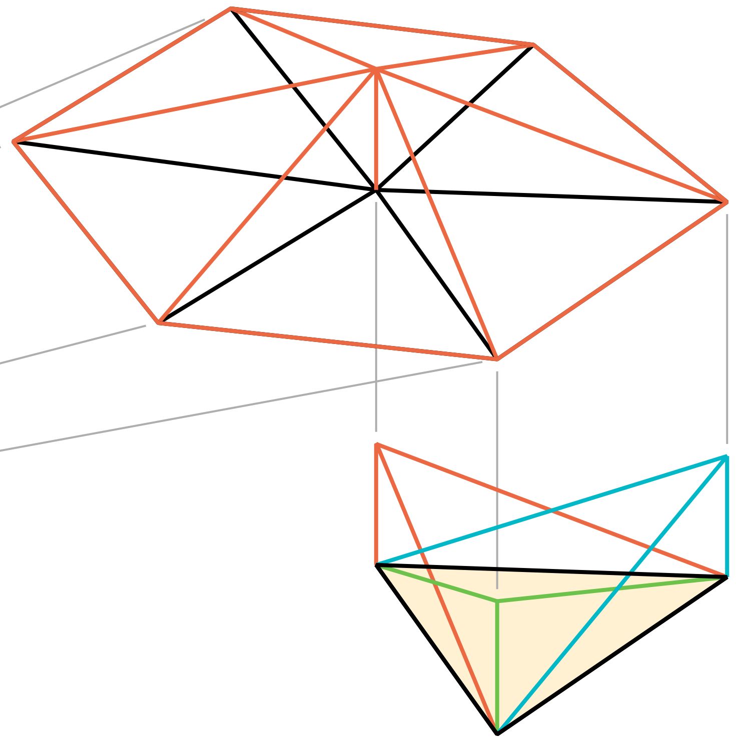
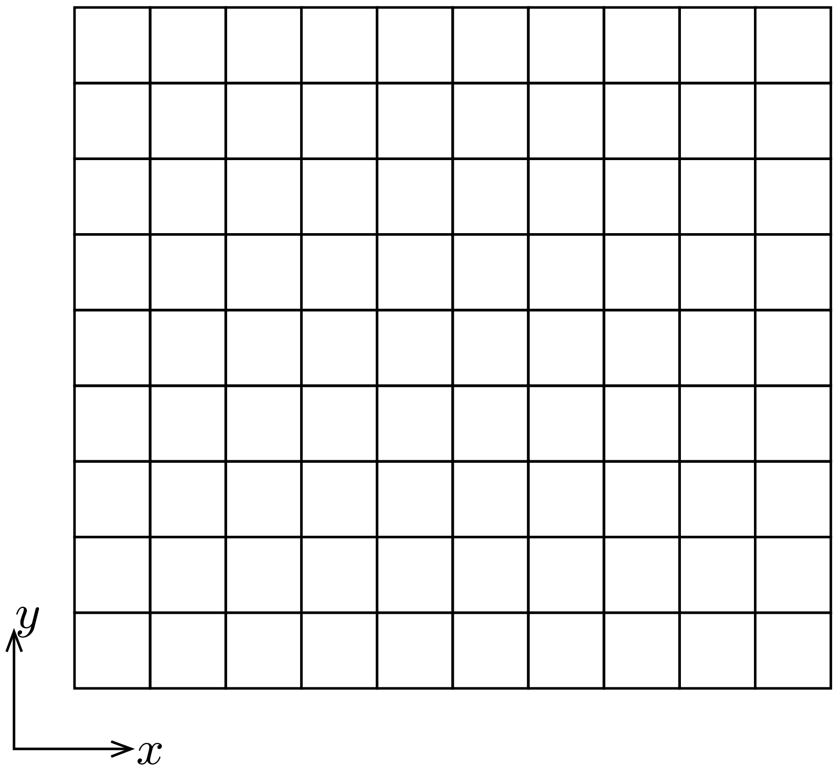


Every element has 3 shape functions:

$$N_i = a_i + b_i x + c_i y$$

2D shape functions on a quadrilateral elements



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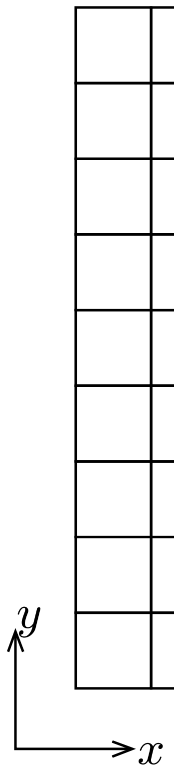

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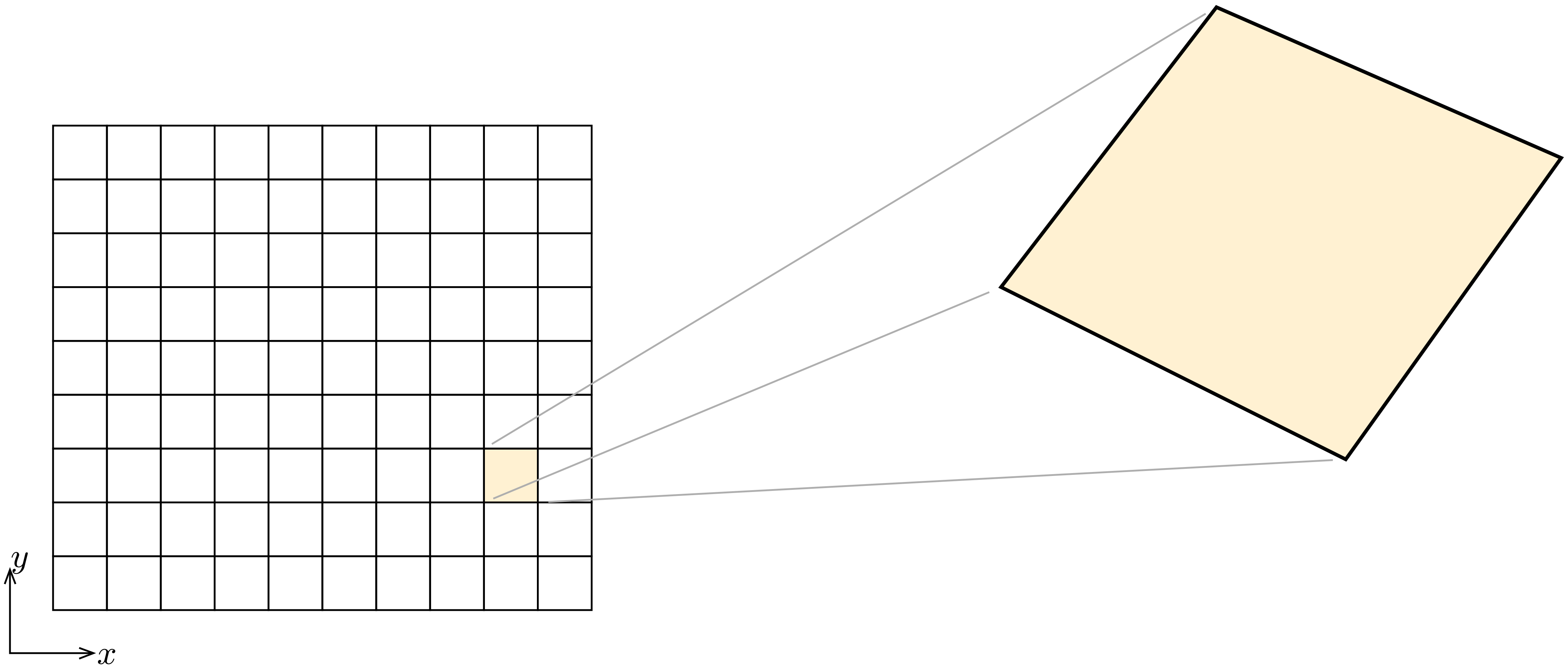

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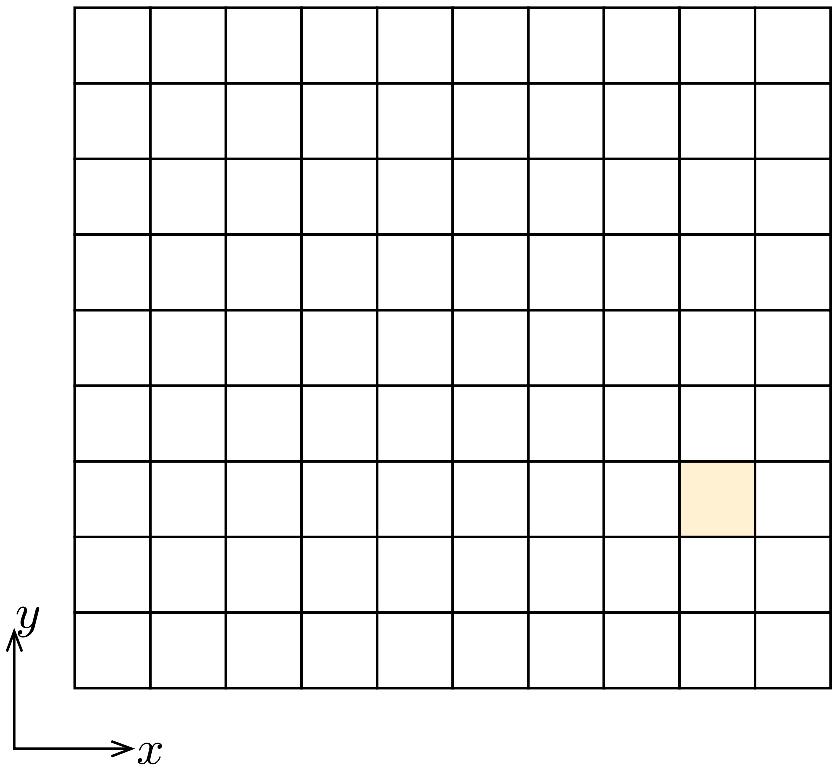
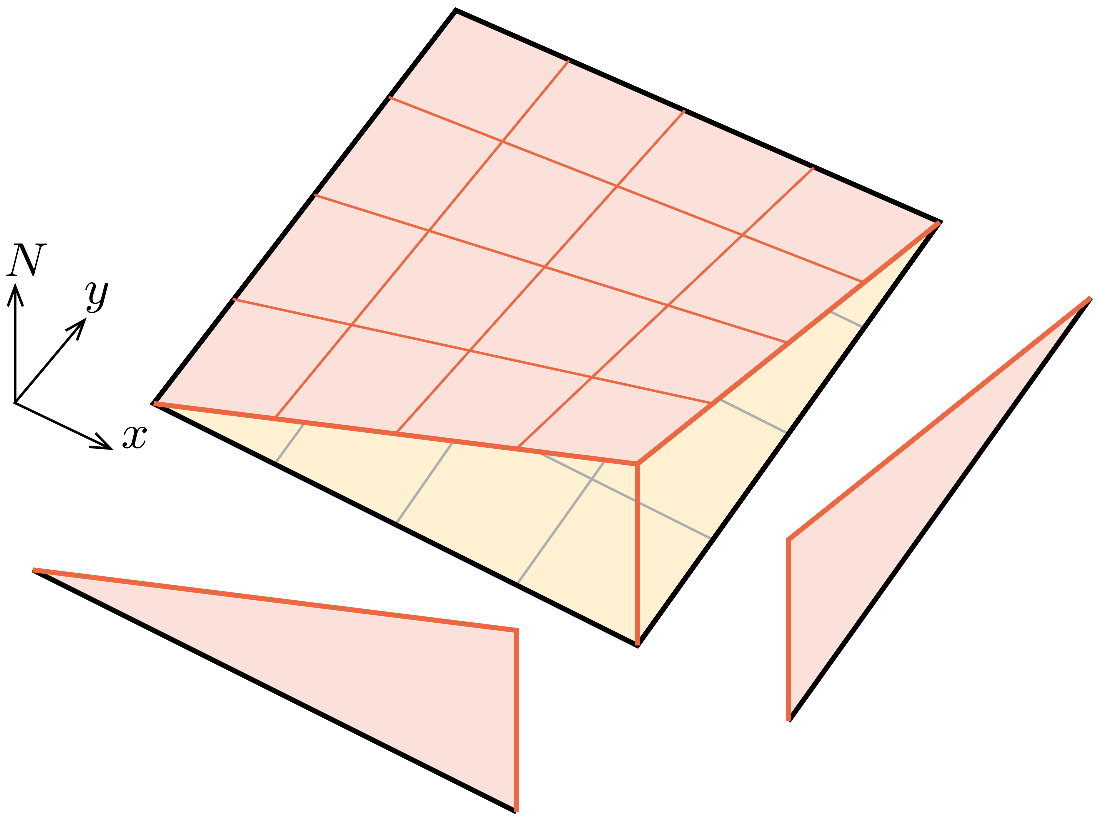
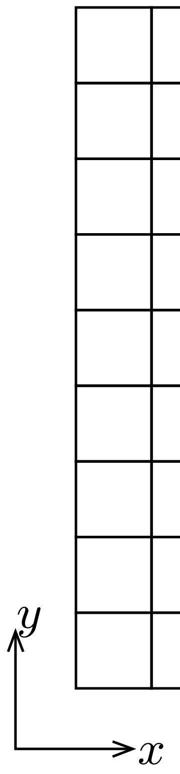


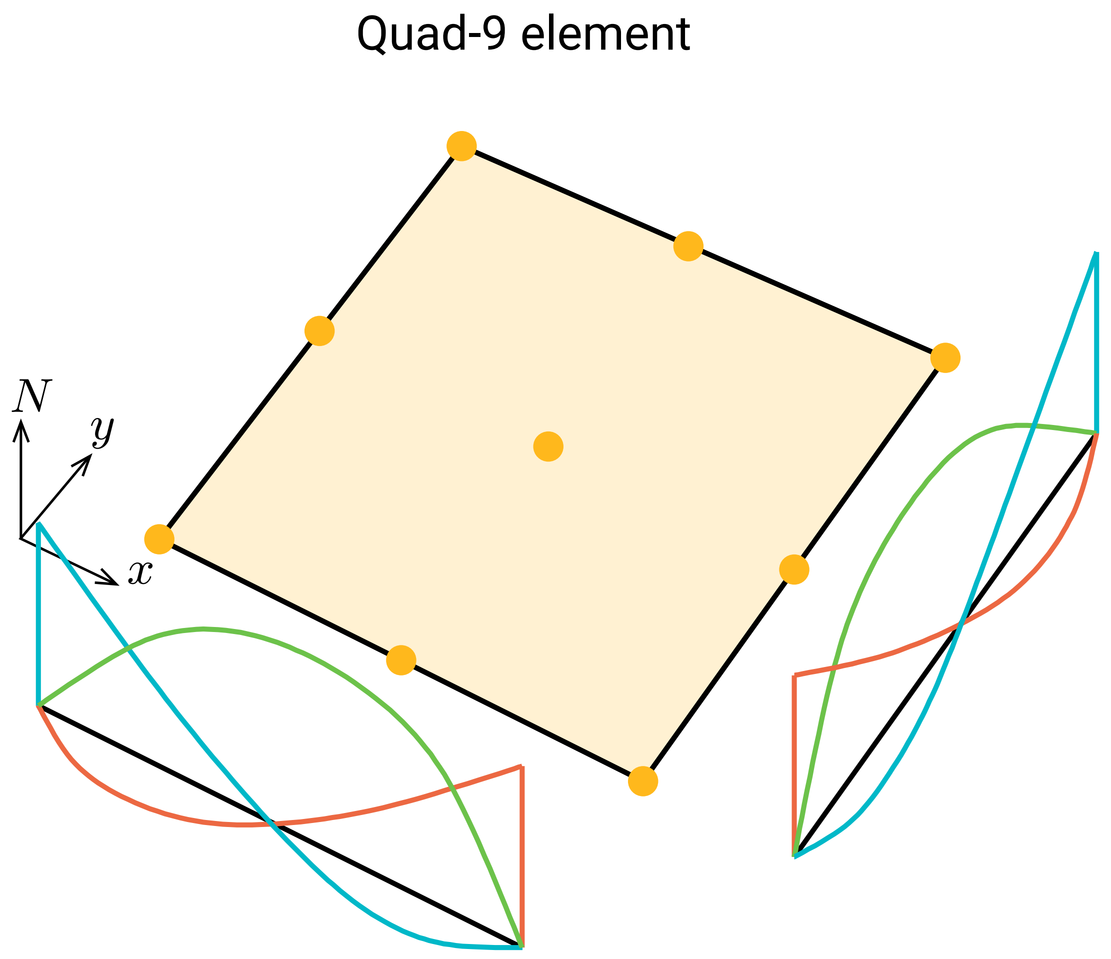
Quad-4 element



$$N_i = a_i + b_i x + c_i y + d_i xy$$

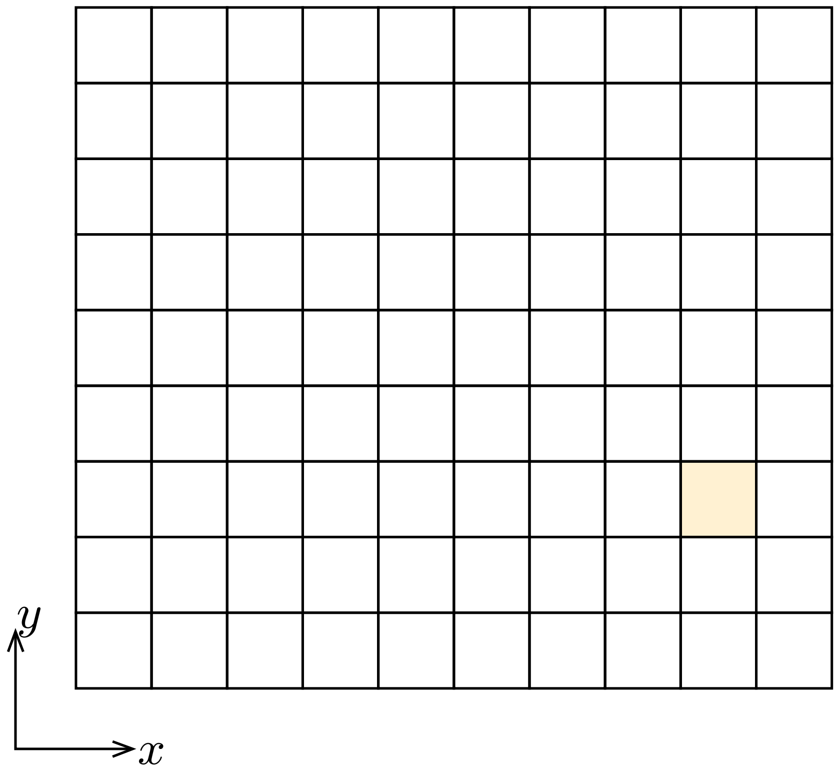
2D shape functions on a quadrilateral elements



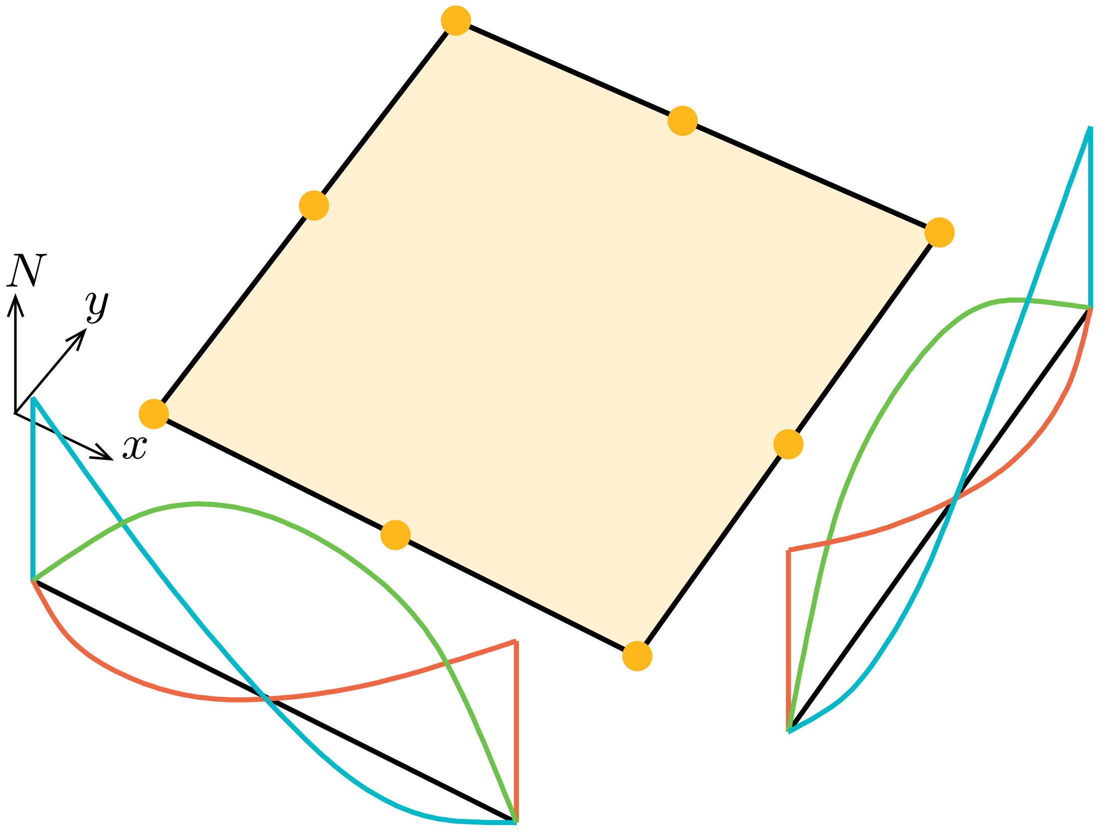


$$N_i = a_i + b_i x + c_i y + d_i x^2 + e_i xy + f_i y^2 + g_i x^2 y + h_i xy^2 + j_i x^2 y^2$$

2D shape functions on a quadrilateral elements



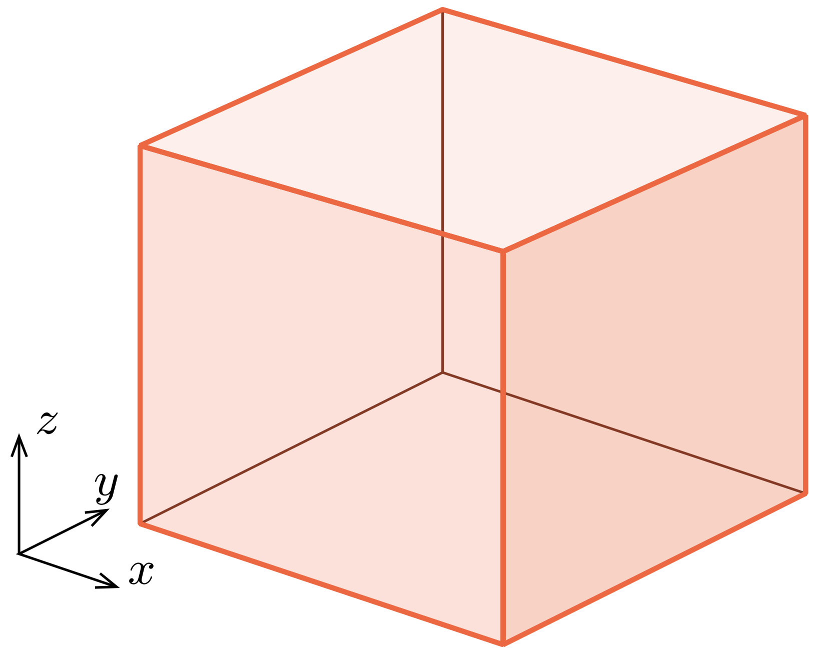
Quad-8 element



$$N_i = a_i + b_i x + c_i y + d_i x^2 + e_i xy + f_i y^2 + g_i x^2 y + h_i xy^2$$

3D elements

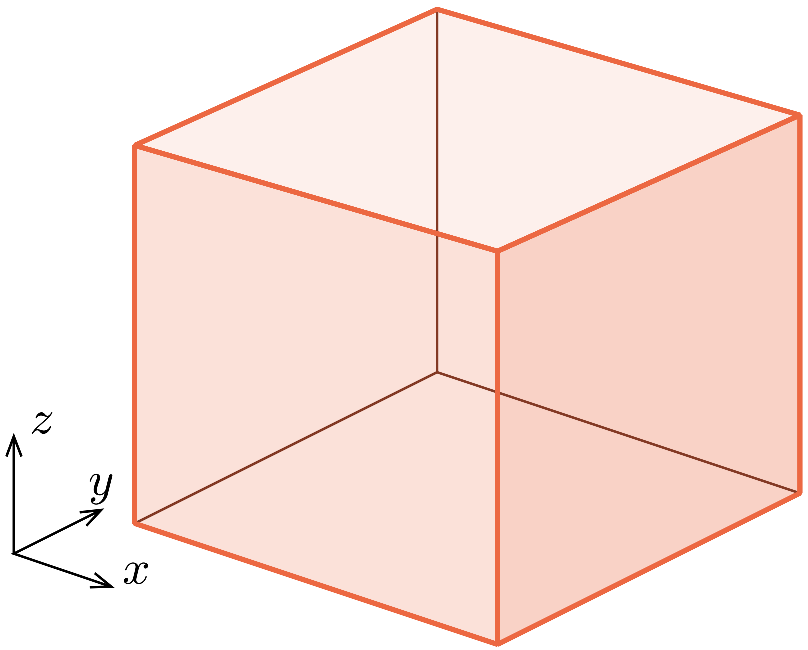
Hexahedral element



$$N_i = a_i + b_i x + c_i y + d_i z + e_i xy + f_i xz + g_i yz + h_i xyz$$

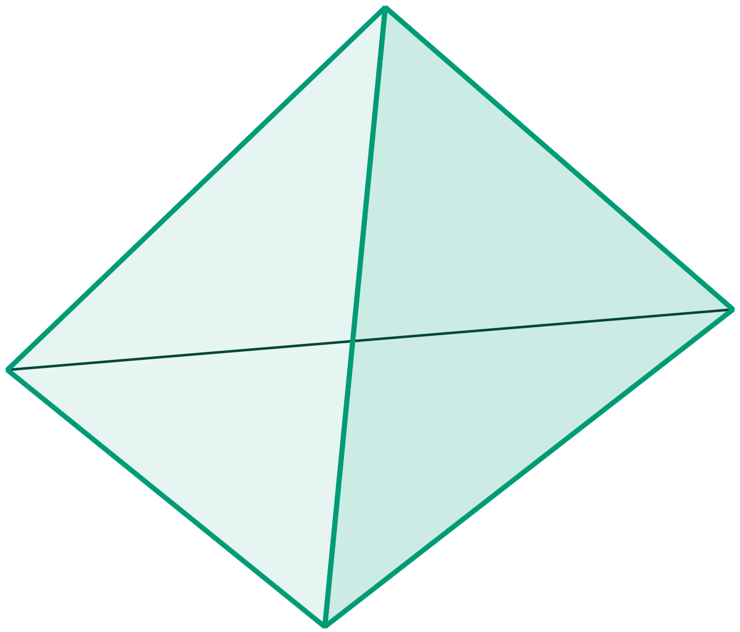
3D elements

Hexahedral element



$$N_i = a_i + b_i x + c_i y + d_i z + e_i xy + f_i xz + g_i yz + h_i xyz$$

Tetrahedral element



$$N_i = a_i + b_i x + c_i y + d_i z$$

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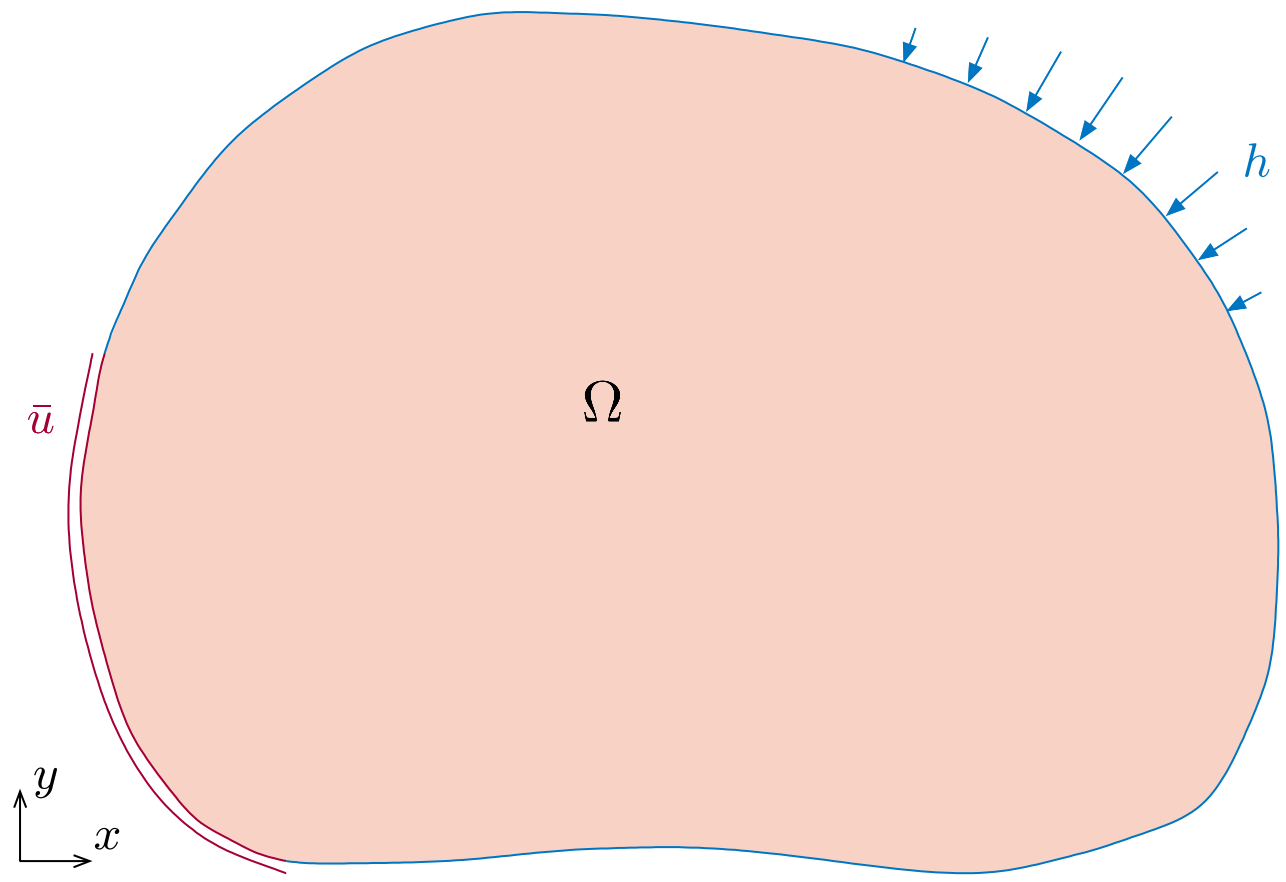
Problem definition in 2D

The Poisson equation:

$$-\nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f \quad \text{on} \quad \Omega$$



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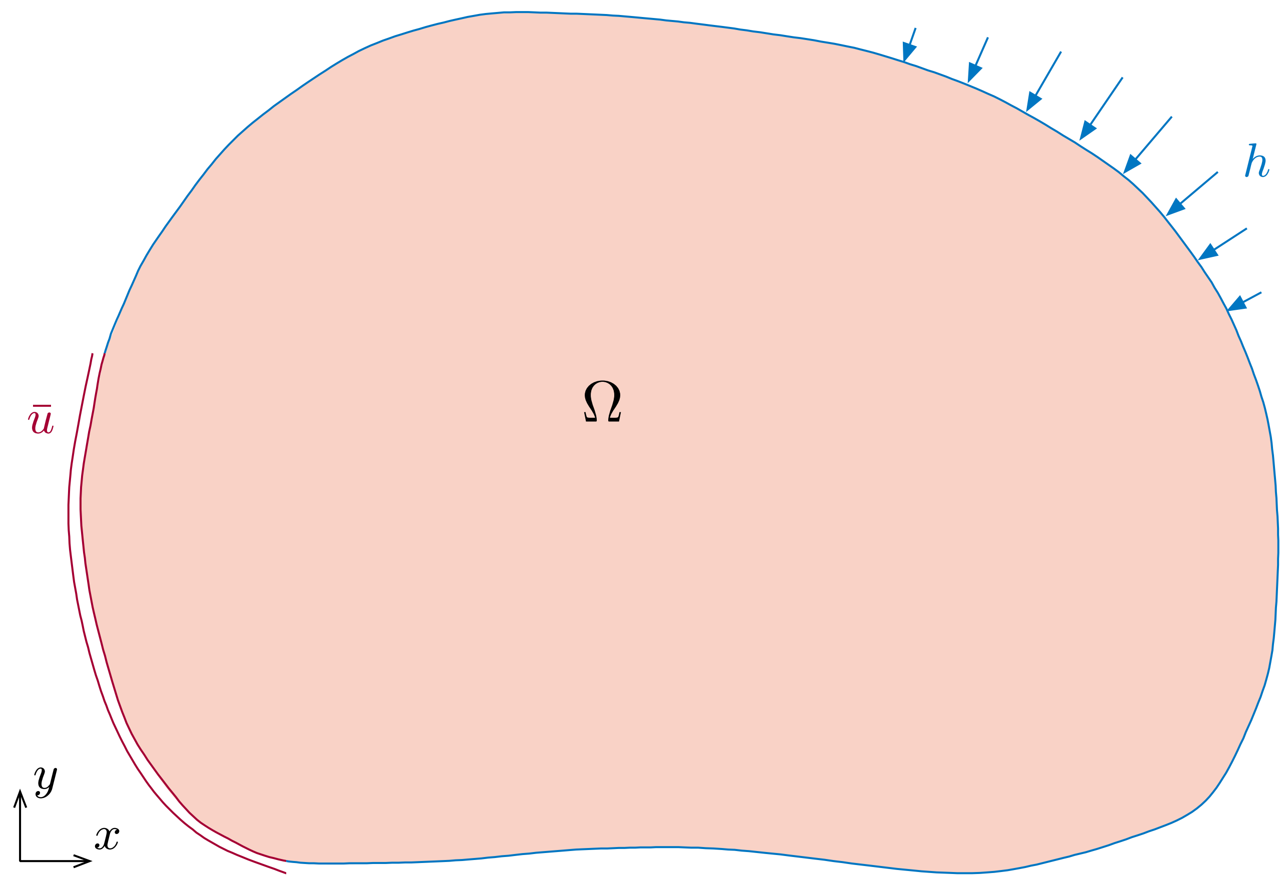
With boundary conditions



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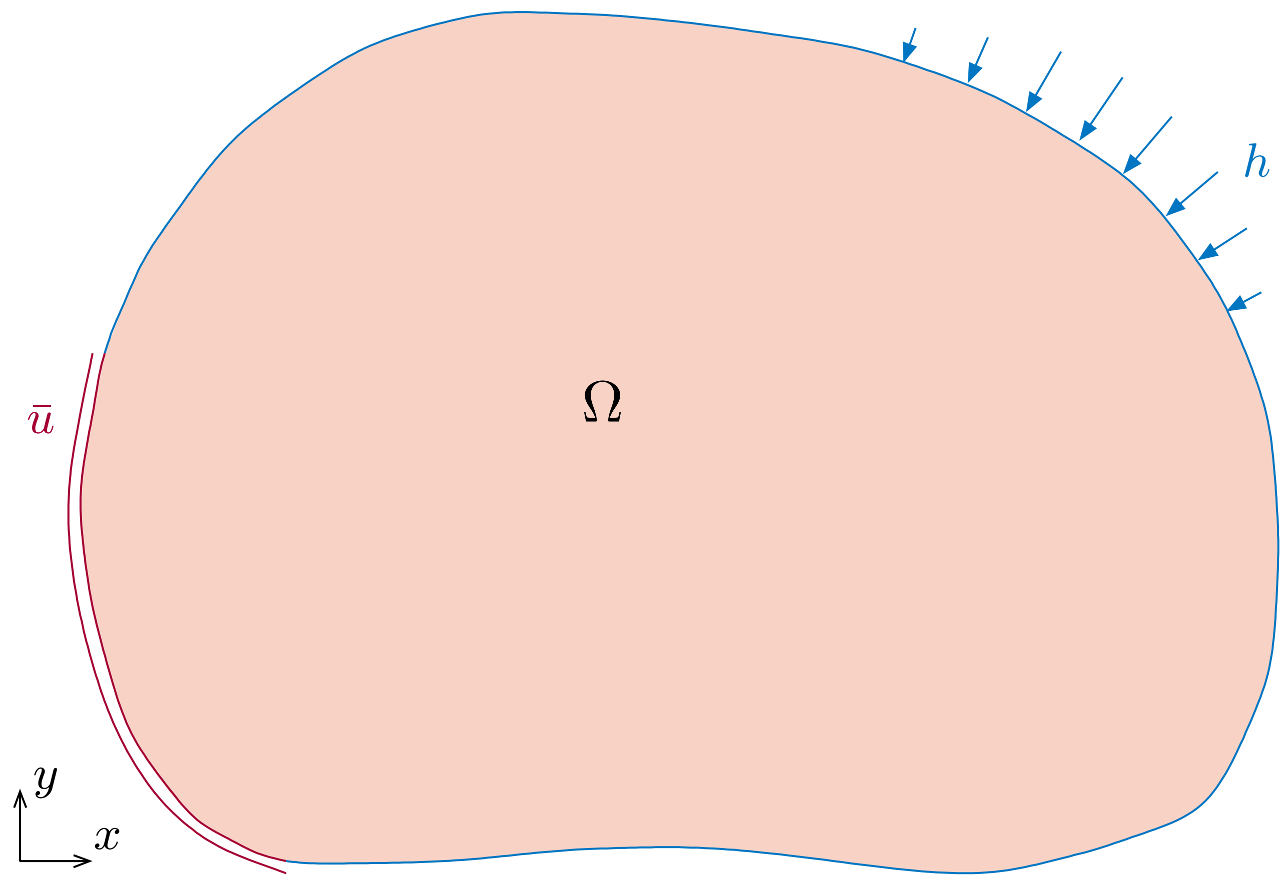
With boundary conditions

$$u=\bar{u}(x,y) \quad \text{on} \quad \Gamma_D$$



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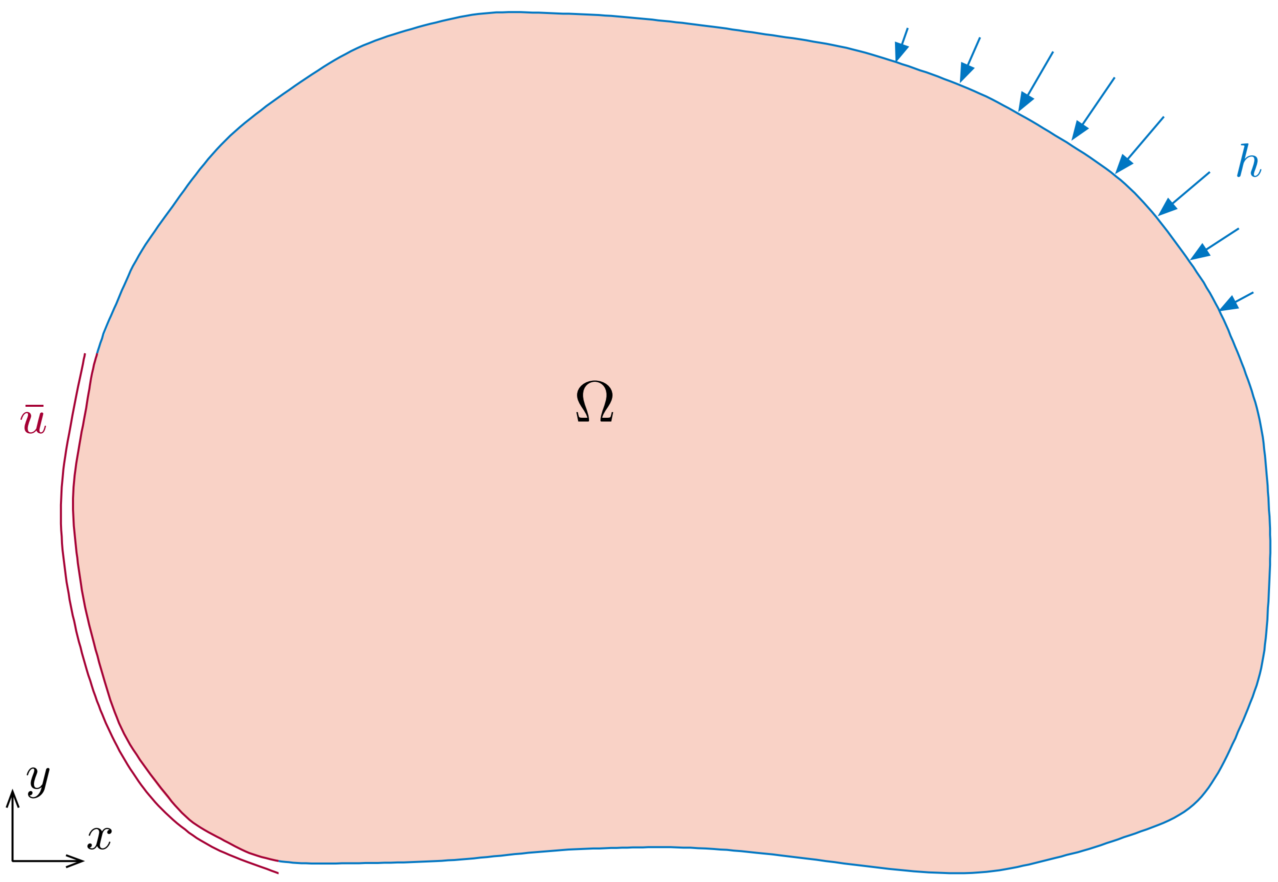
$$\nu\nabla u\cdot\mathbf{n}=h(x,y) \quad \text{on} \quad \Gamma_N$$



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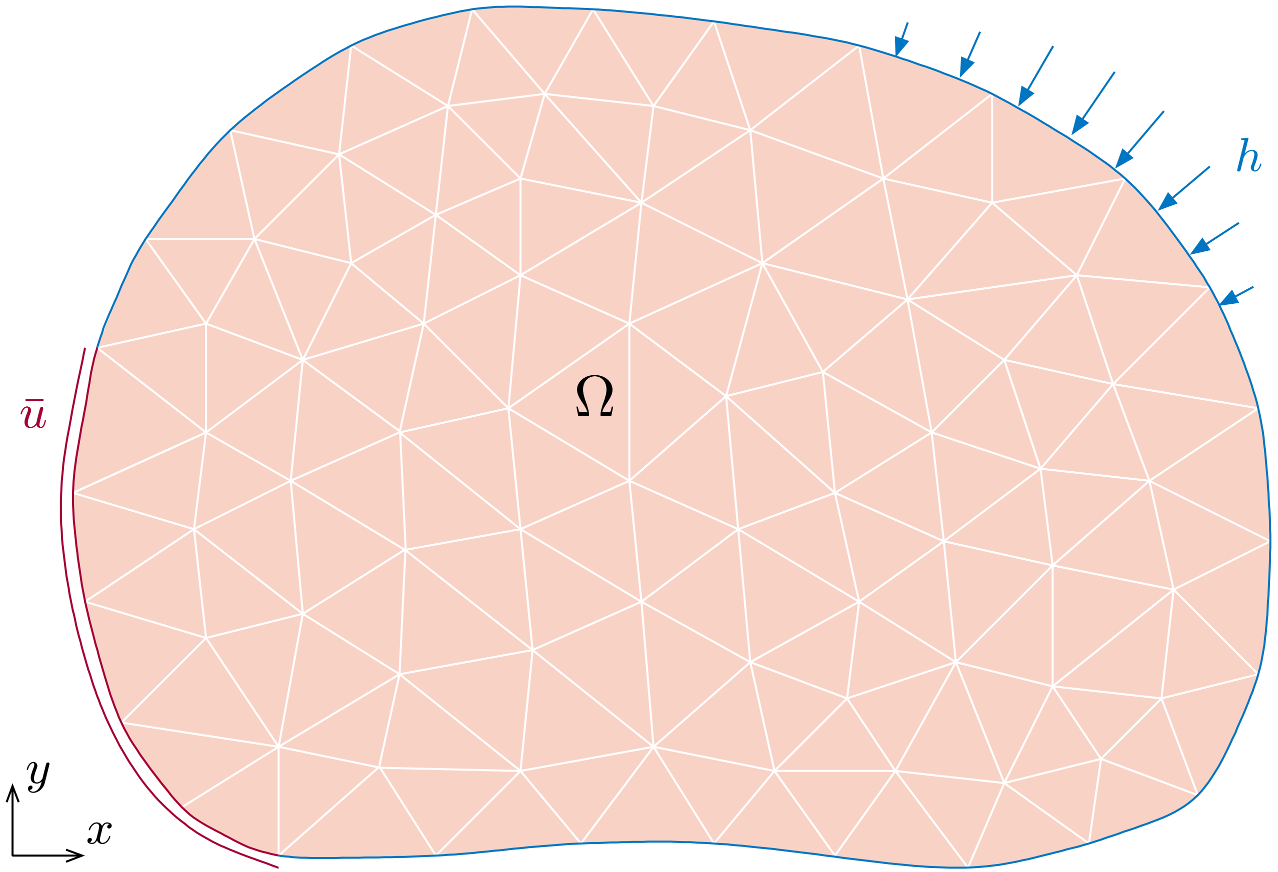


Aim: discretize into a system of equations

$$\mathbf{Ku}=\mathbf{f}$$

Where $\mathbf{u}$ contains approximate values for $u(x,y)$ at the nodes of a finite element mesh

Discretizing the solution in 2D

The Poisson equation:

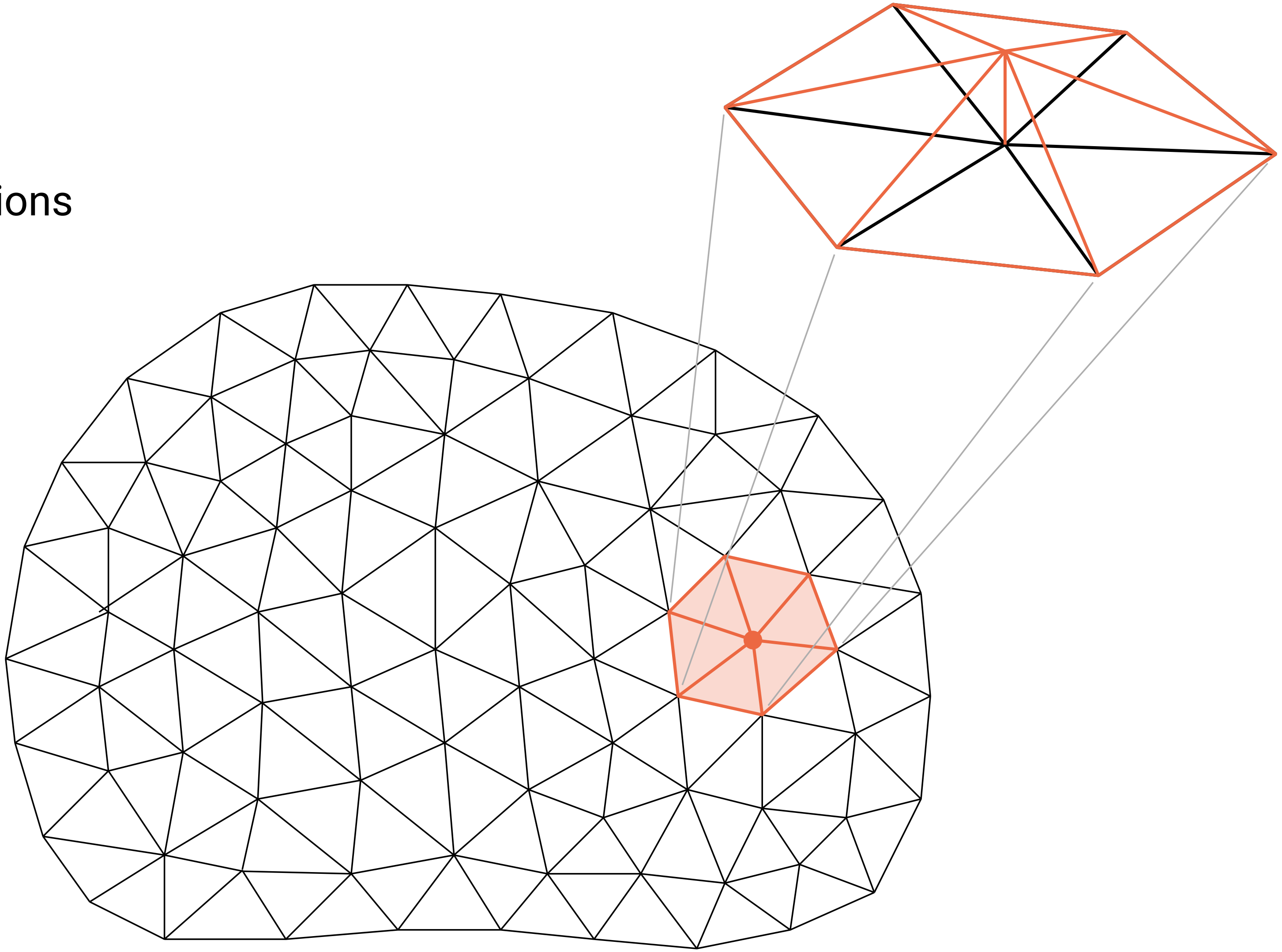
$$-\nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f$$

Approximate u as u^h with 2D shape functions

$$u^h(x, y) = \sum_i N_i(x, y) u_i = \mathbf{N}\mathbf{u}$$

- $\mathbf{u}$ contains nodal values
- $\mathbf{N}$ defines the interpolation

  $\rightarrow$ Find $\mathbf{u}$ such that $u^h \approx u$



Weak form equation in 2D

Weighted residual formulation:

$$-\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f \qquad - \qquad \nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad = \qquad f$$

Weak form equation in 2D

Weighted residual formulation:

$$-\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f \qquad\qquad -\quad {\color{red}w}\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\quad=\quad {\color{red}w}f$$

Weak form equation in 2D

Weighted residual formulation:

$$-\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f \qquad\qquad -\int_\Omega w\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\,\mathrm{d}\Omega=\int_\Omega wf\,\mathrm{d}\Omega$$

Weak form equation in 2D

Weighted residual formulation:

$$-\nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f \qquad \Leftrightarrow \qquad -\int_\Omega w\nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \mathrm{d}\Omega = \int_\Omega w f \, \mathrm{d}\Omega \quad \forall \quad w$$

Weak form equation in 2D

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Integration by parts (with divergence theorem):

$$\int_\Omega w\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}\Omega=-\int_\Omega \nu\nabla w\cdot\nabla u\,\mathrm{d}\Omega+\int_\Gamma w\nu\nabla u\cdot\mathbf{n}\,\mathrm{d}\Gamma \quad \forall \quad w$$

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Weak form equation in 2D

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Substitution:

$$\int_\Omega \nu\nabla w\cdot\nabla u\,\mathrm{d}\Omega-\int_\Gamma w\nu\nabla u\cdot\mathbf{n}\,\mathrm{d}\Gamma=\int_\Omega wf\,\mathrm{d}\Omega \quad \forall \quad w$$

Weak form equation in 2D

Weighted residual formulation:

$$-\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f \qquad \Leftrightarrow \qquad -\int_\Omega w\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\,\mathrm{d}\Omega=\int_\Omega wf\,\mathrm{d}\Omega \quad \forall \quad w$$

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Substitution:

$$\int_\Omega \nu\nabla w\cdot\nabla u\,\mathrm{d}\Omega-\int_\Gamma w\nu\nabla u\cdot\mathbf{n}\,\mathrm{d}\Gamma=\int_\Omega wf\,\mathrm{d}\Omega \quad \forall \quad w$$

With boundary conditions ($w=0$ on Γ_D and $\nu\nabla u\cdot\mathbf{n}=h$ on Γ_N):

$$\int_\Omega \nu\nabla w\cdot\nabla u\,\mathrm{d}\Omega=\int_\Omega wf\,\mathrm{d}\Omega+\int_{\Gamma_N} wh \quad \forall \quad w$$

Discretized form

Weak form equation

$$\int_\Omega \nu \nabla w \cdot \nabla u \, \mathrm{d}\Omega = \int_\Omega w f \, \mathrm{d}\Omega + \int_{\Gamma_N} w h \, \mathrm{d}\Gamma \quad \forall \quad w$$

Introduce discretization:

$$u \leftarrow u^h = \mathbf{N}\mathbf{u}, \qquad w \leftarrow w^h = \mathbf{N}\mathbf{w}, \qquad \mathbf{N} = \begin{bmatrix} N_1 & N_2 & \cdots & N_n \end{bmatrix}$$

$$\nabla u \leftarrow \nabla u^h = \mathbf{B}\mathbf{u}, \qquad \nabla w \leftarrow \nabla w^h = \mathbf{B}\mathbf{w}, \qquad \mathbf{B} = \nabla \mathbf{N} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

Discretized form

Weak form equation

$$\int_{\Omega} \nu \nabla w \cdot \nabla u \, \mathrm{d}\Omega = \int_{\Omega} w f \, \mathrm{d}\Omega + \int_{\Gamma_N} w h \, \mathrm{d}\Gamma \quad \forall \quad w$$

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Substitution gives:

$$\int_{\Omega} \mathbf{B}\mathbf{w} \nu \mathbf{B}\mathbf{u} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{N}\mathbf{w} f \, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{N}\mathbf{w} h \, \mathrm{d}\Gamma \quad \forall \quad \mathbf{w} \qquad \Rightarrow \qquad \int_{\Omega} \mathbf{B}^T \nu \mathbf{B} \, \mathrm{d}\Omega \, \mathbf{u} = \int_{\Omega} \mathbf{N}^T f \, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{N}^T h \, \mathrm{d}\Gamma$$

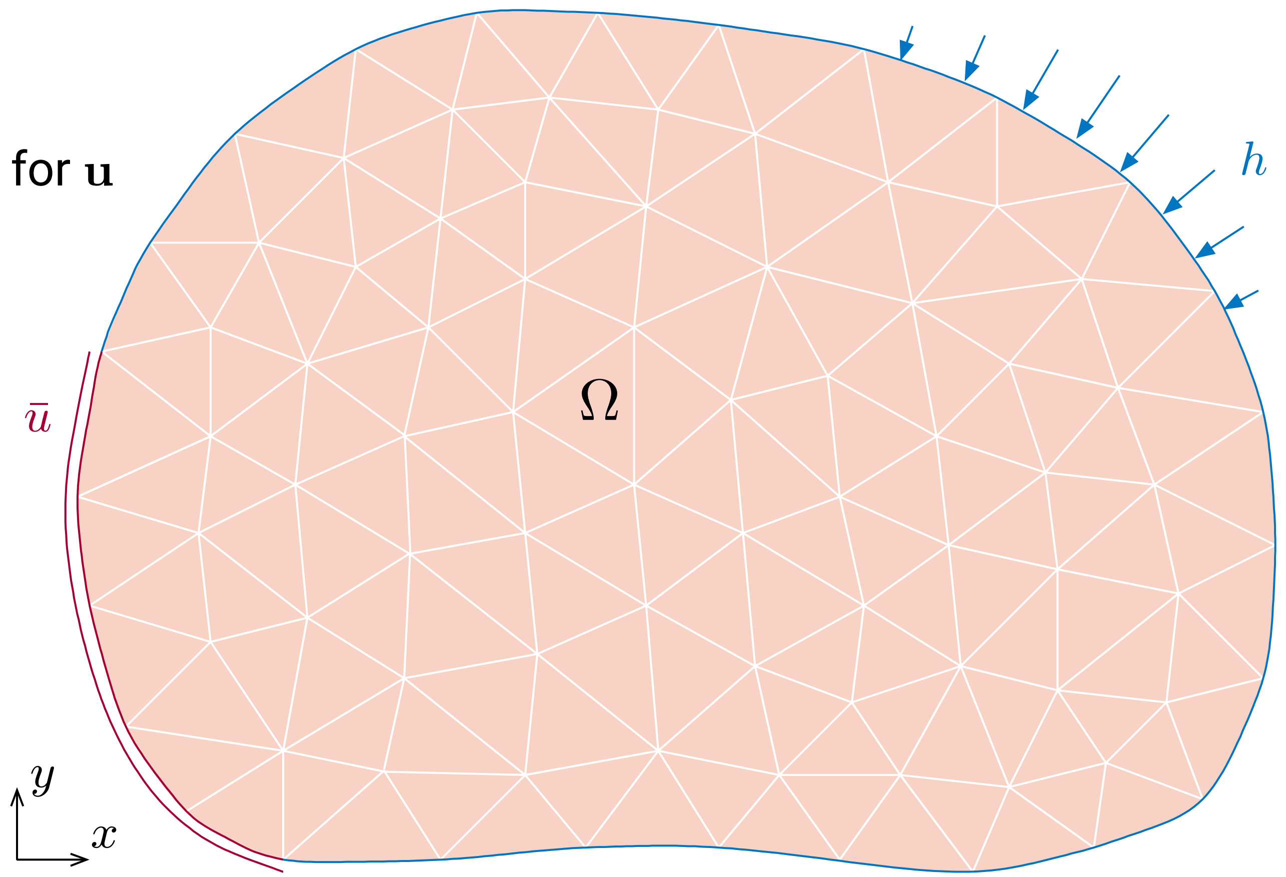
Finding the approximate solution

Discretized form:

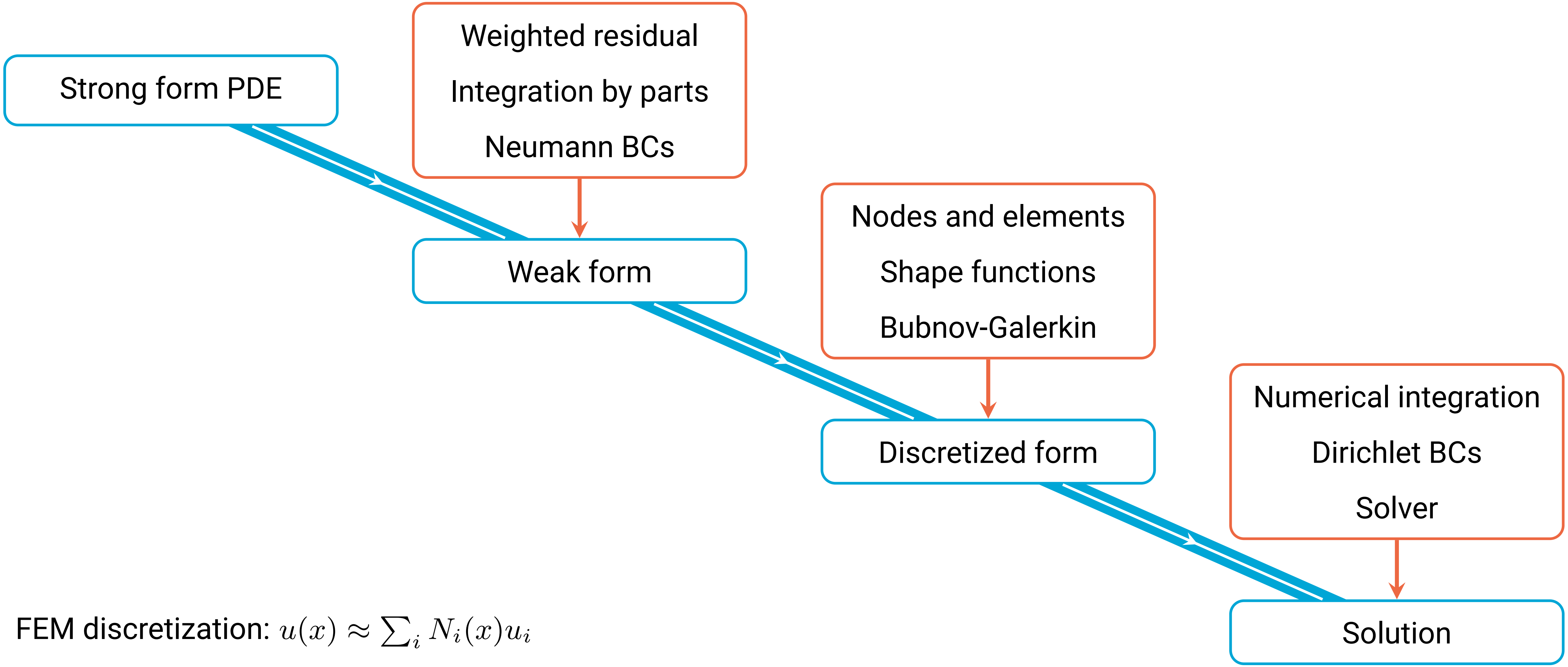
$$\mathbf{K}\mathbf{u} = \mathbf{f} \quad \text{with} \quad \mathbf{K} = \int_\Omega \mathbf{B}^T \nu \mathbf{B}\,\mathrm{d}\Omega \quad \text{and} \quad \mathbf{f} = \int_\Omega \mathbf{N}^T f\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{N}^T h\,\mathrm{d}\Gamma$$

Solving the FE equations finally requires:

- Numerical integration of $\mathbf{K}$ and $\mathbf{f}$
- Constraining $u_i = \bar{u}$ for nodes on Γ_D
- Solving the constrained system of equations for $\mathbf{u}$



Take home message

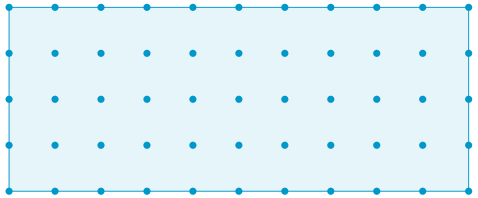


FEM discretization: $u(x) \approx \sum_i N_i(x) u_i$

One more Finite _________ Method

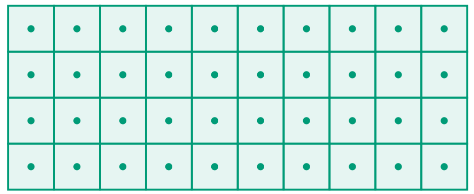
You have seen Finite Difference Method in Q1

- Easiest to implement and understand
- Super efficient for some problems
- Simple geometries and structured grids



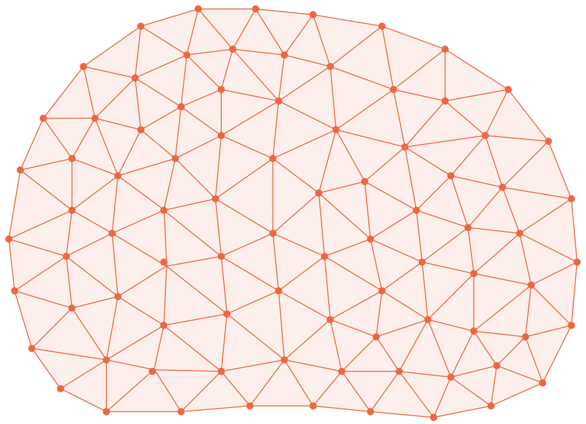
Then the Finite Volume Method (week 2.1)

- Mostly for problems involving flow
- Local conservation is guaranteed



Now the Finite Element Method (week 2.2)

- Originally but not exclusively for solid mechanics
- Straighforward handling of boundary conditions
- Native support for unstructured meshes
- Higher order accuracy with higher order shape functions
- Many other cool possibilities from the choice of shape function



Program for this week

Before Wednesday: Self study

- Book: Poisson equation in 1D + python implementation
- Videos: include additional material

Wednesday: Supported bar problem

- Derive weak form
- Extend python implementation

Friday: Diffusion equation

- Transient problem with FEM
- 2D on non-trivial geometry

Enjoy the week!