CEGM1000
Modelling, Uncertainty and Data for Engineers (round 2)

Week 2.1
Finite Volume Method

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with A LOT of support from Robert Lanzafame and Isabel Slingerland and the rest of MUDEs' wonderful team

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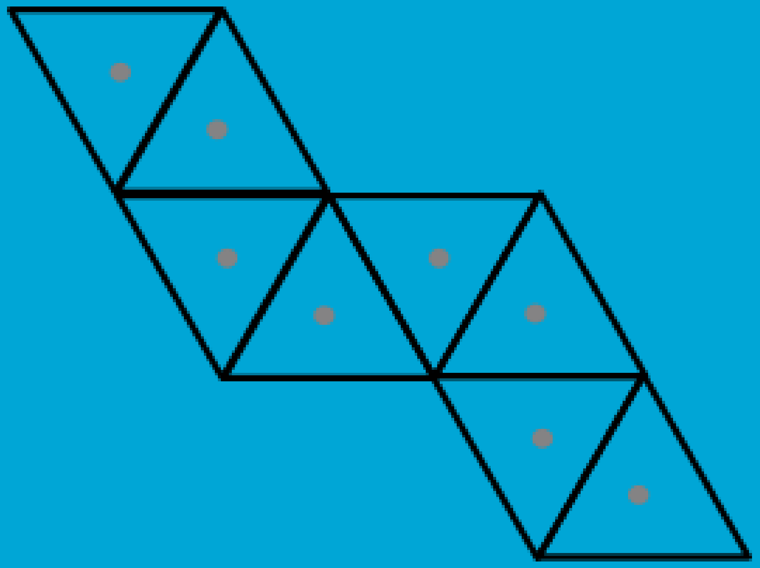

Learning objectives

At the end of this lecture, you should be able to

- Describe the key aspects of the Finite Volume Method and identify the differences with the Finite Difference Method
- Understand PDEs nature to define appropriate numerical schemes
- Apply FVM to structured and unstructured (orthogonal) meshes
- Identify the problems of non-orthogonal meshes

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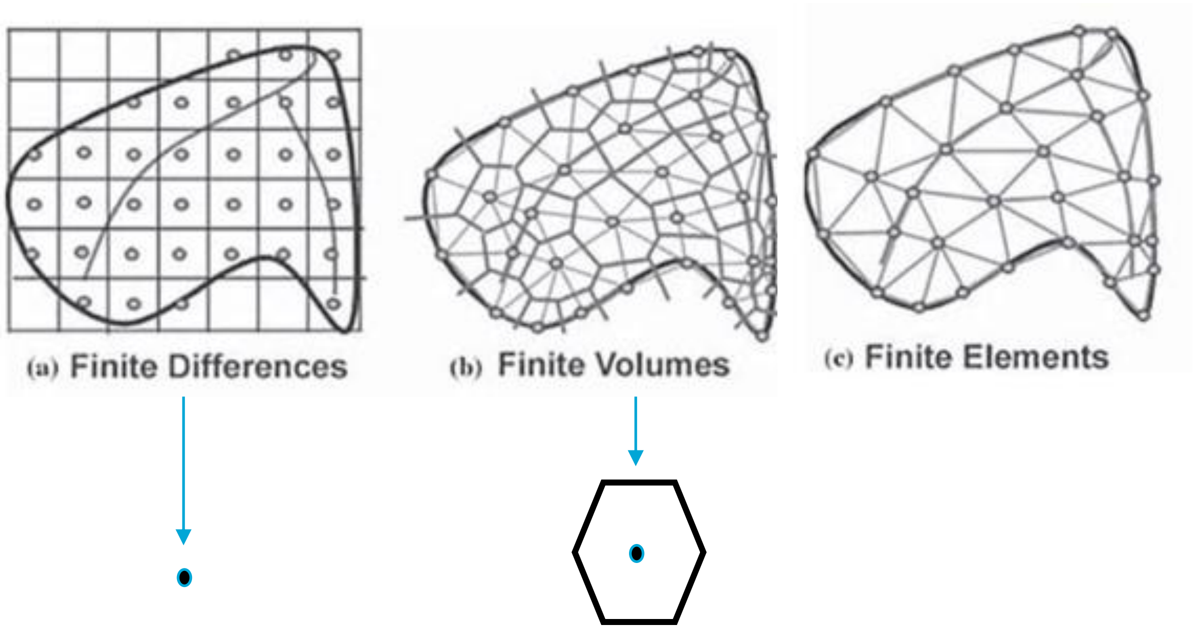
FVM

- Monday: fundamental concepts
- Wednesday: treat advection in 1D and 2D
- Friday: diffusion in unstructured meshes!



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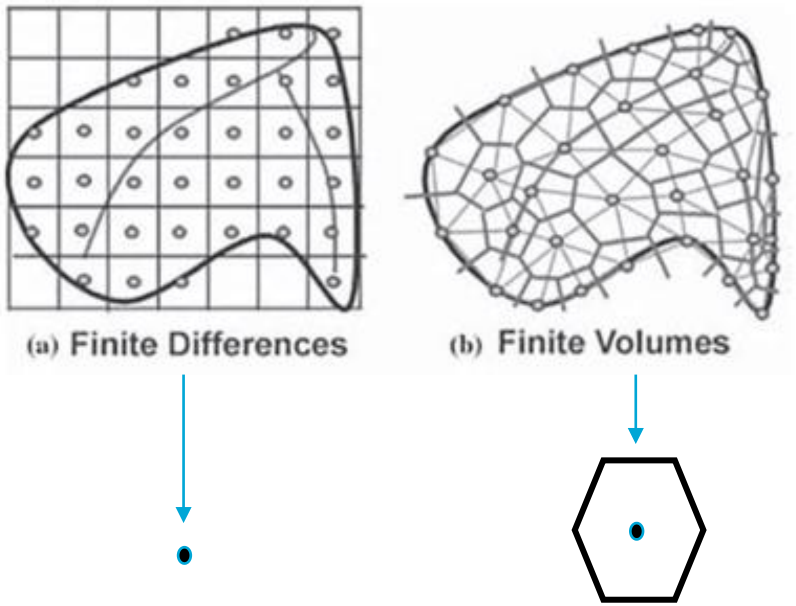
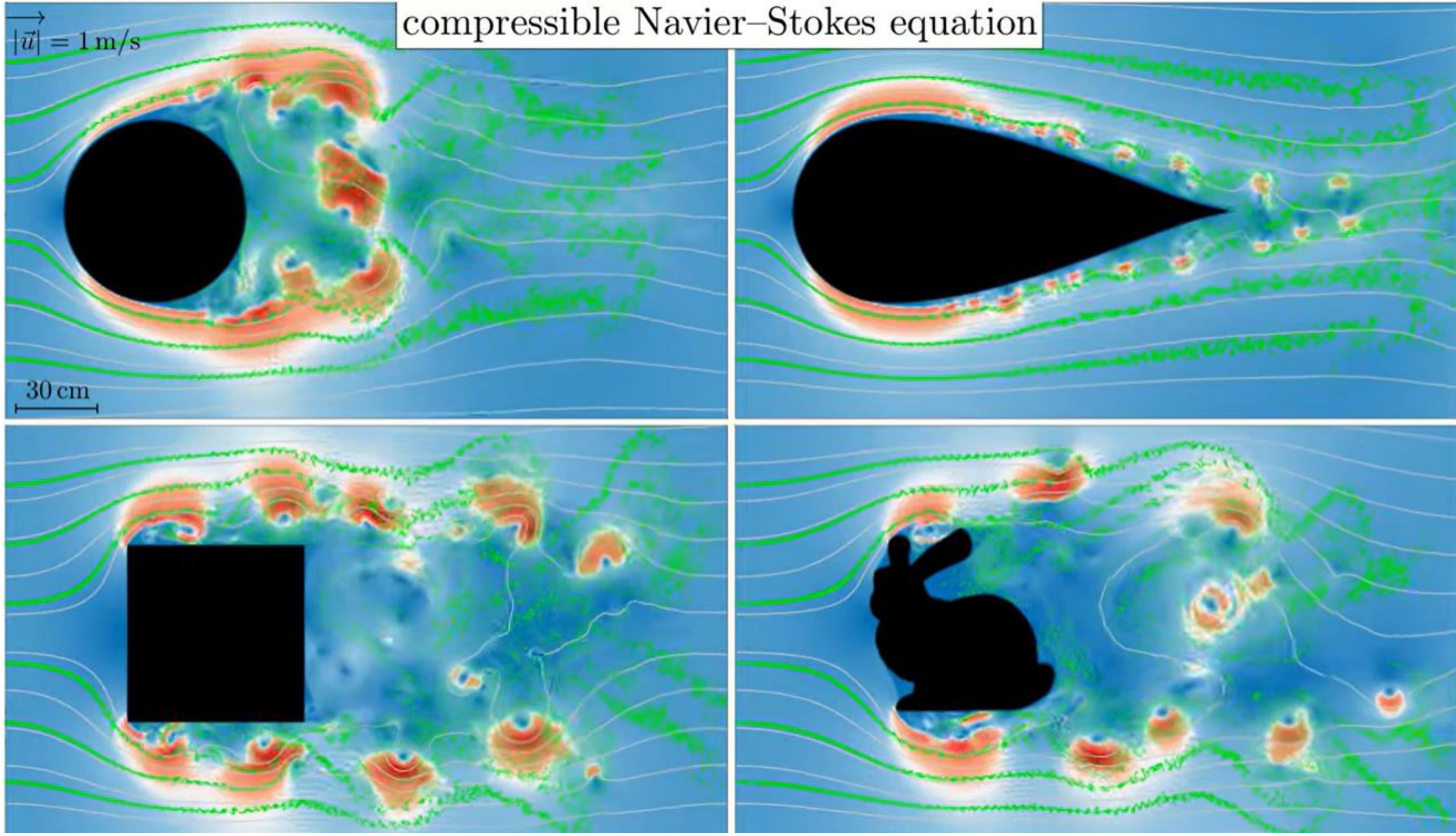
FVM



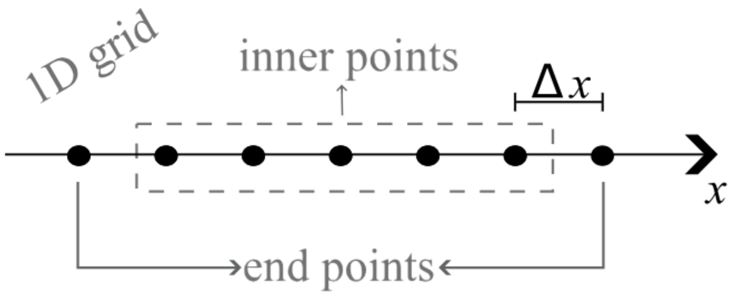


Richters F. “4k Aerodynamics of Different Geometries – 2D Navier Stokes equations”

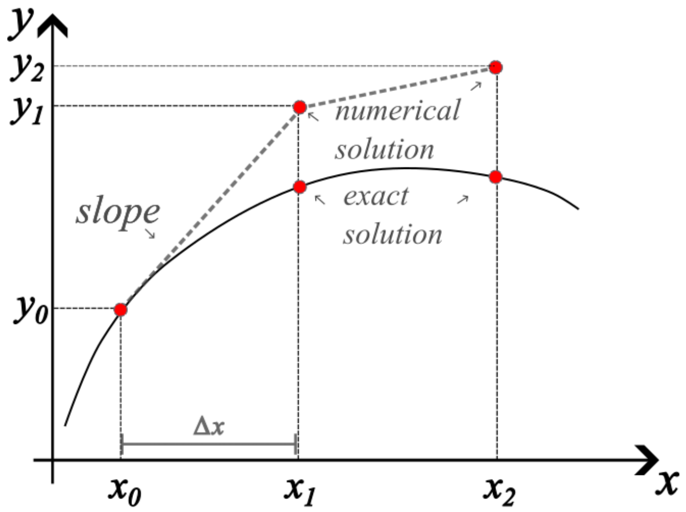
https://www.youtube.com/watch?v=bJX8fVsq5oQ

- Finite Volumes that can vary in shape can accommodate more easily complex geometries! (see rabbit)

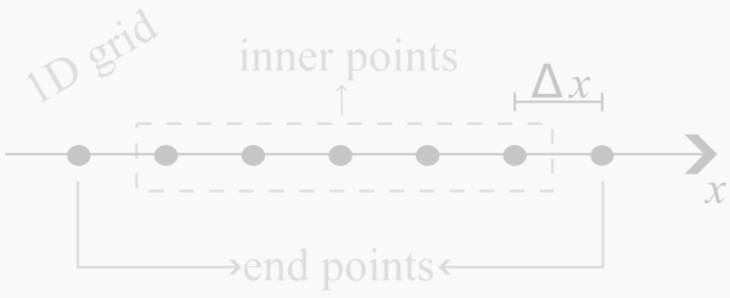
Concepts: FDM vs FVM



- The finite difference method approximates the rate of change of a function using numerical derivatives.



Concepts: FDM vs FVM



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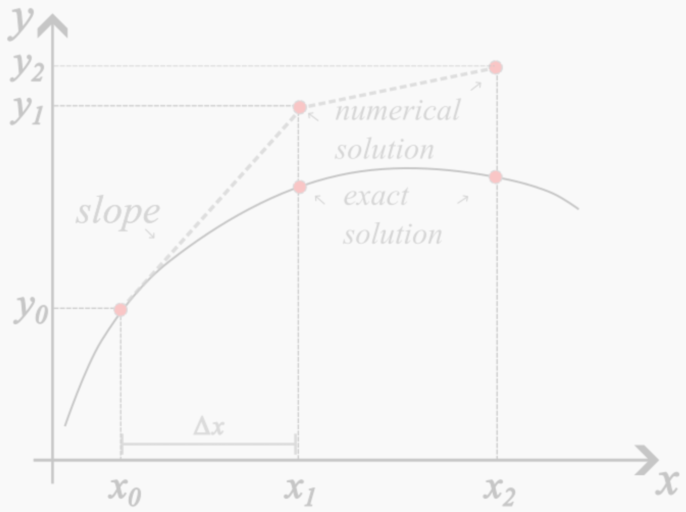
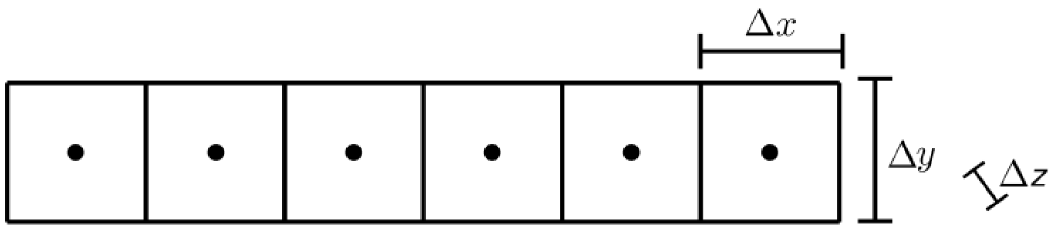
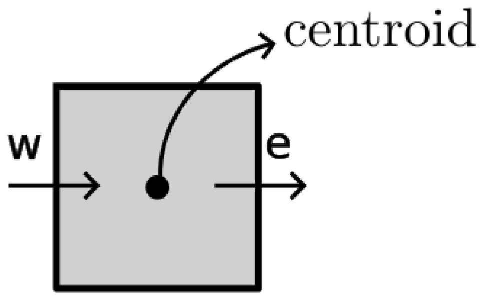




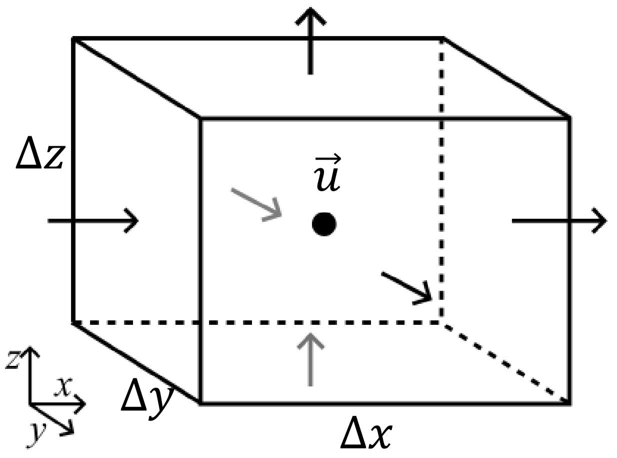
- The finite volume method defines control volumes upon which a quantity is conserved by approximating the fluxes of that quantity in the active surfaces



Mass conservation

$$\frac{\partial(\rho\Delta x\Delta y\Delta z)}{\partial t} =$$

Rate of increase of mass inside the volume



Mass flow rate across the volume surfaces

$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z + \cdots$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

3D mass conservation for compressible fluids

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{u}) = 0$$

3D mass conservation for incompressible fluids

$$\nabla.\vec{u} = 0$$

Transport equation or Advection-Diffusion equation

$$\frac{\partial \rho\phi}{\partial t} + \nabla.(\rho\phi\vec{u}) = \nabla.(D\nabla\phi) + F_{\phi}$$

Rate of increase of ϕ of fluid volume	Net rate of flow of ϕ out of fluid volume **CONVECTION**	Rate of increase of ϕ due to **DIFFUSION**	Rate of increase of ϕ due to sources

We are going to assume a passive scalar quantity = the scalar quantity does not influence the flow

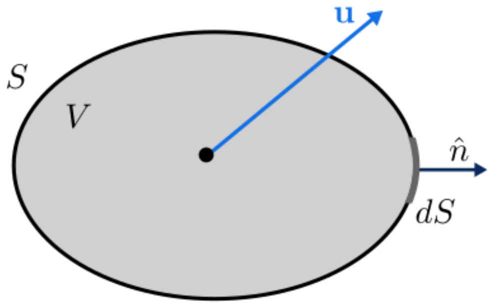
- Unsteady 1D diffusion equation $$\frac{dT}{dt} = \nu\frac{d^2T}{dx^2}$$

- Steady 1D diffusion equation $$\frac{\partial^2 T}{\partial x^2} - \alpha(T - T_s) = 0$$

Finite Volume Method

- Integration over the volume

$$\int_V \frac{\partial \rho\phi}{\partial t} dV \quad + \quad \int_V \nabla.(\rho\phi\vec{u})\, dV \quad = \quad \int_V \nabla.(D\nabla\phi)\, dV \quad + \quad \int_V \mathrm{F}_\phi\, dV$$



- Apply Gauss theorem to transform some volume integrals into surface integrals

$$\int_V \frac{\partial \rho\phi}{\partial t} dV \quad + \quad \int_S \vec{n}.(\rho\phi\vec{u})\, dS \quad = \quad \int_S \vec{n}.(D\nabla\phi)\, dS \quad + \quad \int_V \mathrm{F}_\phi\, dV$$

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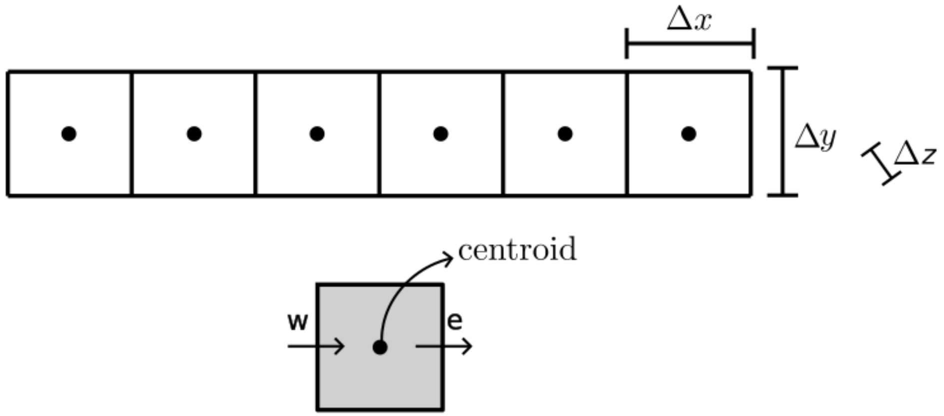
Application to 1D steady diffusion equation

- The following 1D equation describes the steady state solution of the temperature along a pin that sticks out of a furnace. The rest of the pin is exposed to the ambient.



$$\frac{\partial^2 T}{\partial x^2} - \alpha(T - T_s) = 0$$

$$\int_V \cancel{\frac{\partial \rho\phi}{\partial t}} dV \quad + \quad \int_S \cancel{\vec{n}.(\rho\phi\vec{u})\, dS} \quad = \quad \int_S \vec{n}.(D\nabla\phi)\, dS \quad + \quad \int_V \mathrm{F}_\phi\, dV$$

$$\int_{S_w} \vec{n}_w.(\nabla T)\, dS + \int_{S_e} \vec{n}_e.(\nabla T)\, dS - \int_V \alpha(T - T_s)\, dV = 0$$

Application to 1D steady diffusion equation

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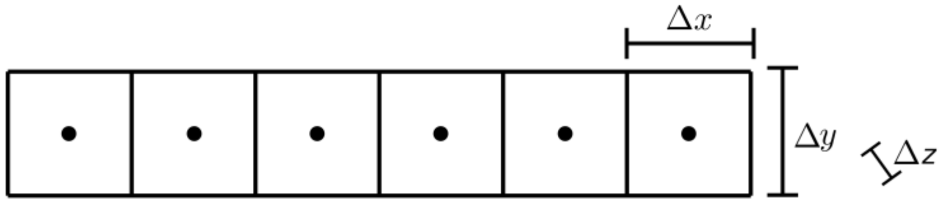
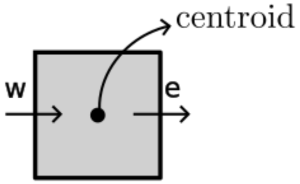


$$\frac{\partial^2 T}{\partial x^2} - \alpha(T - T_s) = 0$$



$$\int_V \frac{\partial \rho\phi}{\partial t} dV + \int_S \vec{n}.(\rho\phi\vec{u})\, dS = \int_S \vec{n}.(D\nabla\phi)\, dS + \int_V \mathrm{F}_\phi\, dV$$

$$\int_{S_w} \vec{n}_w.(\nabla T)\, dS + \int_{S_e} \vec{n}_e.(\nabla T)\, dS - \int_V \alpha(T - T_s)\, dV = 0$$

In 1D the Finite Volume Method ends up giving the exact same algebraic representation as the Finite Difference method using central differences!

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Nature of PDEs

- Elliptic: steady, the perturbation of one inner point affects the rest of points

$$\frac{\partial^2 T}{\partial x^2} - \alpha(T - T_s) = 0$$

- Parabolic: unsteady, the information propagates in all directions at an infinite speed

$$\frac{dT}{dt} = \nu \frac{d^2 T}{dx^2}$$

- Hyperbolic: generally unsteady, the information propagates at a finite speed

$$\frac{\partial \phi}{\partial t} + c\,\frac{\partial \phi}{\partial x} = 0$$

The physical behaviour of PDEs influences the adequate numerical schemes to solve them!

Exercise: apply FVM to the convection equation

- Don't worry about the boundaries now!

$$\frac{\partial \phi}{\partial t} + c\,\frac{\partial \phi}{\partial x} = 0$$

$$\int_V \nabla \cdot \mathbf{r}\, dV = \int_S \mathbf{n} \cdot \mathbf{r}\, dS$$

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1D Convection equation

- 1 initial condition (state of ϕ at t=0 along the entire domain) and 1 boundary condition at the boundary from which the information is being propagated.

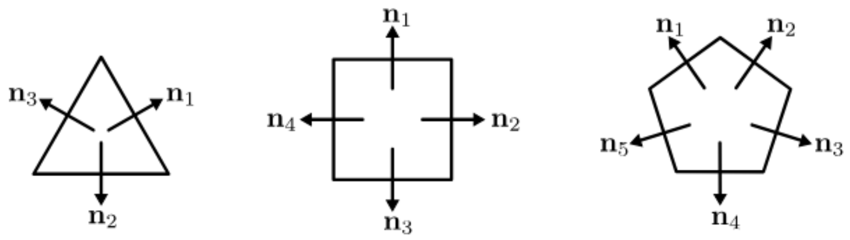
$$\frac{\partial \phi}{\partial t} + c\,\frac{\partial \phi}{\partial x} = 0$$

The information being propagated comes from “upwind” and central differences does not care about flow direction (propagation of information direction).

To be tested on Wednesday ☺

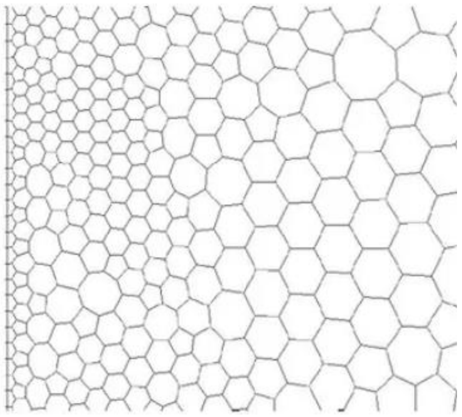
Upwind schemes solve this issue but at a cost: numerical (fake) diffusion.

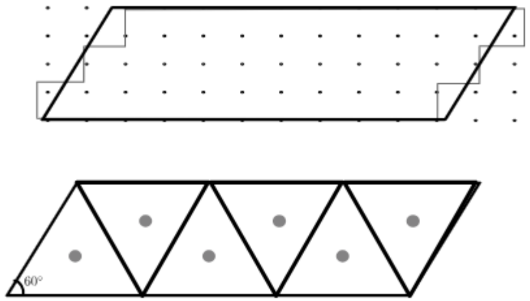
Unstructured mesh



- A mesh can be composed of any shape (or combination of shapes).

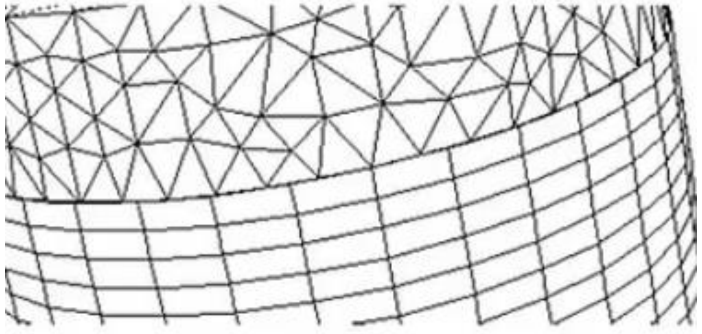
- The quantity ϕ is defined at the centroids





FDM

FVM



Diffusion in unstructured meshes will be treated on Friday ☺

Diffusion: orthogonal vs non-orthogonal meshes

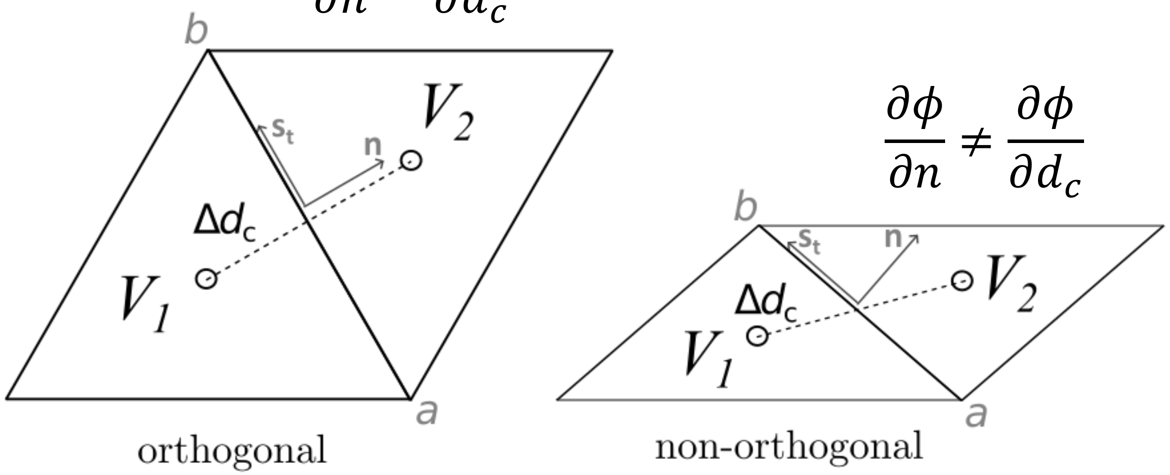
- In orthogonal meshes, constructed for example with equilateral triangles. the central difference is an adequate approach to compute the flux due to diffusion

- For non-orthogonal meshes, a new term is required and surrounding $\phi's$

$$\int_S \vec{n}.(D\nabla\phi)\, dS = D\frac{\partial\phi}{\partial n}\Delta S$$

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial d_c}$$

$$\frac{\partial\phi}{\partial n} \neq \frac{\partial\phi}{\partial d_c}$$



Takeaways

- Advection dominated problems require different treatment than diffusion dominated problems
- FVM is based on conservation laws and fluxes trough the volume’s surfaces. This is possible using Gauss’s theorem.
- FVM is especially adequate for complex geometries and is based on intuitive physical mechanisms
- Orthogonal meshes limit the geometries that can be represented. Non-orthogonal meshes can represent them but come with downsides regarding complexity of implementation, computational time and added error sources
- The degree of non-orthogonality should be kept low